

SUBLINEAR OPERATORS WITH ROUGH KERNEL GENERATED BY FRACTIONAL INTEGRALS AND COMMUTATORS ON GENERALIZED VANISHING LOCAL MORREY SPACES

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ABSTRACT. In this paper, we consider the norm inequalities for sublinear operators with rough kernel generated by fractional integrals and commutators on generalized local Morrey spaces and on generalized vanishing local Morrey spaces including their weak versions under generic size conditions which are satisfied by most of the operators in harmonic analysis, respectively. As an example to the conditions of these theorems are satisfied, we can consider the Marcinkiewicz operator.

1. INTRODUCTION

The classical Morrey spaces $L_{p,\lambda}$ have been introduced by Morrey in [43] to study the local behavior of solutions of second order elliptic partial differential equations(PDEs). Later, there are many applications of Morrey space to the Navier-Stokes equations (see [40]), the Schrödinger equations (see [52]) and the elliptic problems with discontinuous coefficients (see [6, 20, 47]).

Let $B = B(x_0, r_B)$ denote the ball with the center x_0 and radius r_B . For a given measurable set E , we also denote the Lebesgue measure of E by $|E|$. For any given $\Omega \subseteq \mathbb{R}^n$ and $0 < p < \infty$, denote by $L_p(\Omega)$ the spaces of all functions f satisfying

$$\|f\|_{L_p(\Omega)} = \left(\int_{\Omega} |f(x)|^p dx \right)^{\frac{1}{p}} < \infty.$$

We recall the definition of classical Morrey spaces $L_{p,\lambda}$ as

$$L_{p,\lambda}(\mathbb{R}^n) = \left\{ f : \|f\|_{L_{p,\lambda}(\mathbb{R}^n)} = \sup_{x \in \mathbb{R}^n, r > 0} r^{-\frac{\lambda}{p}} \|f\|_{L_p(B(x,r))} < \infty \right\},$$

where $f \in L_p^{loc}(\mathbb{R}^n)$, $0 \leq \lambda \leq n$ and $1 \leq p < \infty$.

Note that $L_{p,0} = L_p(\mathbb{R}^n)$ and $L_{p,n} = L_{\infty}(\mathbb{R}^n)$. If $\lambda < 0$ or $\lambda > n$, then $L_{p,\lambda} = \Theta$, where Θ is the set of all functions equivalent to 0 on \mathbb{R}^n .

We also denote by $WL_{p,\lambda} \equiv WL_{p,\lambda}(\mathbb{R}^n)$ the weak Morrey space of all functions $f \in WL_p^{loc}(\mathbb{R}^n)$ for which

$$\|f\|_{WL_{p,\lambda}} \equiv \|f\|_{WL_{p,\lambda}(\mathbb{R}^n)} = \sup_{x \in \mathbb{R}^n, r > 0} r^{-\frac{\lambda}{p}} \|f\|_{WL_p(B(x,r))} < \infty,$$

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where $WL_p(B(x, r))$ denotes the weak L_p -space of measurable functions f for which

$$\begin{aligned} \|f\|_{WL_p(B(x, r))} &\equiv \|f\chi_{B(x, r)}\|_{WL_p(\mathbb{R}^n)} \\ &= \sup_{t>0} t |\{y \in B(x, r) : |f(y)| > t\}|^{1/p} \\ &= \sup_{0 < t \leq |B(x, r)|} t^{1/p} (f\chi_{B(x, r)})^*(t) < \infty, \end{aligned}$$

where g^* denotes the non-increasing rearrangement of a function g .

Throughout the paper we assume that $x \in \mathbb{R}^n$ and $r > 0$ and also let $B(x, r)$ denotes the open ball centered at x of radius r , $B^C(x, r)$ denotes its complement and $|B(x, r)|$ is the Lebesgue measure of the ball $B(x, r)$ and $|B(x, r)| = v_n r^n$, where $v_n = |B(0, 1)|$. It is known that $L_{p, \lambda}(\mathbb{R}^n)$ is an extension of $L_p(\mathbb{R}^n)$ in the sense that $L_{p, 0} = L_p(\mathbb{R}^n)$.

Morrey has stated that many properties of solutions to PDEs can be attributed to the boundedness of some operators on Morrey spaces. For the boundedness of the Hardy–Littlewood maximal operator, the fractional integral operator and the Calderón–Zygmund singular integral operator on these spaces, we refer the readers to [1, 10, 49]. For the properties and applications of classical Morrey spaces, see [11, 12, 19, 20] and references therein.

The study of the operators of harmonic analysis in vanishing Morrey space, in fact has been almost not touched. A version of the classical Morrey space $L_{p, \lambda}(\mathbb{R}^n)$ where it is possible to approximate by "nice" functions is the so called vanishing Morrey space $VM_{p, \lambda}(\mathbb{R}^n)$ has been introduced by Vitanza in [64] and has been applied there to obtain a regularity result for elliptic PDEs. This is a subspace of functions in $L_{p, \lambda}(\mathbb{R}^n)$, which satisfies the condition

$$\lim_{r \rightarrow 0} \sup_{\substack{x \in \mathbb{R}^n \\ 0 < t < r}} t^{-\frac{\lambda}{p}} \|f\|_{L_p(B(x, t))} = 0.$$

Later in [65] Vitanza has proved an existence theorem for a Dirichlet problem, under weaker assumptions than in [41] and a $W^{3,2}$ regularity result assuming that the partial derivatives of the coefficients of the highest and lower order terms belong to vanishing Morrey spaces depending on the dimension. Also Ragusa has proved a sufficient condition for commutators of fractional integral operators to belong to vanishing Morrey spaces $VL_{p, \lambda}(\mathbb{R}^n)$ ([50, 51]). For the properties and applications of vanishing Morrey spaces, see also [7]. It is known that, there is no research regarding boundedness of the sublinear operators with rough kernel on vanishing Morrey spaces.

Maximal functions and singular integrals play a key role in harmonic analysis since maximal functions could control crucial quantitative information concerning the given functions, despite their larger size, while singular integrals, Hilbert transform as it's prototype, recently intimately connected with PDEs, operator theory and other fields.

Let $f \in L^{loc}(\mathbb{R}^n)$. The Hardy–Littlewood(H–L) maximal operator M is defined by

$$Mf(x) = \sup_{t>0} |B(x, t)|^{-1} \int_{B(x, t)} |f(y)| dy.$$

Let \bar{T} be a standard Calderón–Zygmund(C–Z) singular integral operator, briefly a C–Z operator, i.e., a linear operator bounded from $L_2(\mathbb{R}^n)$ to $L_2(\mathbb{R}^n)$ taking

all infinitely continuously differentiable functions f with compact support to the functions $f \in L_1^{loc}(\mathbb{R}^n)$ represented by

$$\overline{T}f(x) = p.v. \int_{\mathbb{R}^n} k(x-y)f(y) dy \quad x \notin \text{supp}f.$$

Such operators have been introduced in [15]. Here k is a C-Z kernel [25]. Chiarenza and Frasca [10] have obtained the boundedness of H-L maximal operator M and C-Z operator \overline{T} on $L_{p,\lambda}(\mathbb{R}^n)$. It is also well known that H-L maximal operator M and C-Z operator \overline{T} play an important role in harmonic analysis (see [23, 38, 59, 60, 62]). Also, the theory of the C-Z operator is one of the important achievements of classical analysis in the last century, which has many important applications in Fourier analysis, complex analysis, operator theory and so on.

Let $f \in L_1^{loc}(\mathbb{R}^n)$. The fractional maximal operator M_α and the fractional integral operator (also known as the Riesz potential) \overline{T}_α are defined by

$$M_\alpha f(x) = \sup_{t>0} |B(x,t)|^{-1+\frac{\alpha}{n}} \int_{B(x,t)} |f(y)| dy \quad 0 \leq \alpha < n$$

$$\overline{T}_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy \quad 0 < \alpha < n.$$

It is well known that M_α and \overline{T}_α play an important role in harmonic analysis (see [60, 62]).

An early impetus to the study of fractional integrals originated from the problem of fractional derivation, see e.g. [46]. Besides its contributions to harmonic analysis, fractional integrals also play an essential role in many other fields. The H-L Sobolev inequality about fractional integral is still an indispensable tool to establish time-space estimates for the heat semigroup of nonlinear evolution equations, for some of this work, see e.g. [33]. In recent times, the applications to Chaos and Fractal have become another motivation to study fractional integrals, see e.g. [35]. It is well known that \overline{T}_α is bounded from L_p to L_q , where $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n}$ and $1 < p < \frac{n}{\alpha}$.

Spanne (published by Peetre [49]) and Adams [1] have studied boundedness of the fractional integral operator \overline{T}_α on $L_{p,\lambda}(\mathbb{R}^n)$. Their results, can be summarized as follows.

Theorem 1. (*Spanne, but published by Peetre [49]*) Let $0 < \alpha < n$, $1 < p < \frac{n}{\alpha}$, $0 < \lambda < n - \alpha p$. Moreover, let $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n}$ and $\frac{\lambda}{p} = \frac{\mu}{q}$. Then for $p > 1$ the operator \overline{T}_α is bounded from $L_{p,\lambda}$ to $L_{q,\lambda}$ and for $p = 1$ the operator \overline{T}_α is bounded from $L_{1,\lambda}$ to $WL_{q,\lambda}$.

Theorem 2. (*Adams [1]*) Let $0 < \alpha < n$, $1 < p < \frac{n}{\alpha}$, $0 < \lambda < n - \alpha p$ and $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n-\lambda}$. Then for $p > 1$ the operator \overline{T}_α is bounded from $L_{p,\lambda}$ to $L_{q,\lambda}$ and for $p = 1$ the operator \overline{T}_α is bounded from $L_{1,\lambda}$ to $WL_{q,\lambda}$.

Recall that, for $0 < \alpha < n$,

$$M_\alpha f(x) \leq \nu_n^{\frac{\alpha}{n}-1} \overline{T}_\alpha(|f|)(x)$$

holds (see [34], Remark 2.1). Hence Theorems 1 and 2 also imply boundedness of the fractional maximal operator M_α , where ν_n is the volume of the unit ball on \mathbb{R}^n .

Suppose that S^{n-1} is the unit sphere on \mathbb{R}^n ($n \geq 2$) equipped with the normalized Lebesgue measure $d\sigma$. Let $\Omega \in L_s(S^{n-1})$ with $1 < s \leq \infty$ be homogeneous of degree zero. We define $s' = \frac{s}{s-1}$ for any $s > 1$. Suppose that $T_{\Omega,\alpha}$, $\alpha \in (0, n)$ represents a linear or a sublinear operator, which satisfies that for any $f \in L_1(\mathbb{R}^n)$ with compact support and $x \notin \text{supp} f$

$$(1.1) \quad |T_{\Omega,\alpha}f(x)| \leq c_0 \int_{\mathbb{R}^n} \frac{|\Omega(x-y)|}{|x-y|^{n-\alpha}} |f(y)| dy,$$

where c_0 is independent of f and x .

We point out that the condition (1.1) in the case of $\Omega \equiv 1$, $\alpha = 0$ has been introduced by Soria and Weiss in [57]. The condition (1.1) is satisfied by many interesting operators in harmonic analysis, such as fractional maximal operator, fractional integral operator (Riesz potential), fractional Marcinkiewicz operator and so on (see [37], [57] for details).

In 1971, Muckenhoupt and Wheeden [45] defined the fractional integral operator with rough kernel $\overline{T}_{\Omega,\alpha}$ by

$$\overline{T}_{\Omega,\alpha}f(x) = \int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^{n-\alpha}} f(y) dy \quad 0 < \alpha < n$$

and a related fractional maximal operator with rough kernel $M_{\Omega,\alpha}$ is given by

$$M_{\Omega,\alpha}f(x) = \sup_{t>0} |B(x,t)|^{-1+\frac{\alpha}{n}} \int_{B(x,t)} |\Omega(x-y)| |f(y)| dy \quad 0 \leq \alpha < n,$$

where $\Omega \in L_s(S^{n-1})$ with $1 < s \leq \infty$ is homogeneous of degree zero on \mathbb{R}^n and $\overline{T}_{\Omega,\alpha}$ satisfies the condition (1.1).

If $\alpha = 0$, then $M_{\Omega,0} \equiv M_{\Omega}$ H-L maximal operator with rough kernel. It is obvious that when $\Omega \equiv 1$, $M_{1,\alpha} \equiv M_{\alpha}$ and $\overline{T}_{1,\alpha} \equiv \overline{T}_{\alpha}$ are the fractional maximal operator and the fractional integral operator, respectively.

In recent years, the mapping properties of $\overline{T}_{\Omega,\alpha}$ on some kinds of function spaces have been studied in many papers (see [9], [17], [18], [45] for details). In particular, the boundedness of $\overline{T}_{\Omega,\alpha}$ in Lebesgue spaces has been obtained.

Lemma 1. ([9, 17, 44]) *Let $0 < \alpha < n$, $1 < p < \frac{n}{\alpha}$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. If $\Omega \in L_s(S^{n-1})$, $s > \frac{n}{n-\alpha}$, then we have*

$$\|\overline{T}_{\Omega,\alpha}f\|_{L_q} \leq C \|f\|_{L_p}.$$

Corollary 1. *Under the assumptions of Lemma 1, the operator $M_{\Omega,\alpha}$ is bounded from $L_p(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$. Moreover, we have*

$$\|M_{\Omega,\alpha}f\|_{L_q} \leq C \|f\|_{L_p}.$$

Proof. Set

$$\tilde{T}_{|\Omega|,\alpha}(|f|)(x) = \int_{\mathbb{R}^n} \frac{|\Omega(x-y)|}{|x-y|^{n-\alpha}} |f(y)| dy \quad 0 < \alpha < n,$$

where $\Omega \in L_s(S^{n-1})$ ($s > 1$) is homogeneous of degree zero on \mathbb{R}^n . It is easy to see that, for $\tilde{T}_{|\Omega|,\alpha}$, Lemma 1 is also hold. On the other hand, for any $t > 0$, we have

$$\begin{aligned} \tilde{T}_{|\Omega|,\alpha}(|f|)(x) &\geq \int_{B(x,t)} \frac{|\Omega(x-y)|}{|x-y|^{n-\alpha}} |f(y)| dy \\ &\geq \frac{1}{t^{n-\alpha}} \int_{B(x,t)} |\Omega(x-y)| |f(y)| dy. \end{aligned}$$

Taking the supremum for $t > 0$ on the inequality above, we get

$$M_{\Omega,\alpha} f(x) \leq C_{n,\alpha}^{-1} \tilde{T}_{|\Omega|,\alpha}(|f|)(x) \quad C_{n,\alpha} = |B(0,1)|^{\frac{n-\alpha}{n}}.$$

□

In 1976, Coifman, Rocherberg and Weiss [13] introduced the commutator generated by \overline{T}_Ω and a local integrable function b :

$$(1.2) \quad [b, \overline{T}_\Omega]f(x) \equiv b(x)\overline{T}_\Omega f(x) - \overline{T}_\Omega(bf)(x) = p.v. \int_{\mathbb{R}^n} [b(x) - b(y)] \frac{\Omega(x-y)}{|x-y|^n} f(y) dy.$$

Sometimes, the commutator defined by (1.2) is also called the commutator in Coifman-Rocherberg-Weiss's sense, which has its root in the complex analysis and harmonic analysis (see [13]).

Let b be a locally integrable function on \mathbb{R}^n , then we shall define the commutators generated by fractional integral operators with rough kernel and b as follows.

$$[b, \overline{T}_{\Omega,\alpha}]f(x) \equiv b(x)\overline{T}_{\Omega,\alpha} f(x) - \overline{T}_{\Omega,\alpha}(bf)(x) = p.v. \int_{\mathbb{R}^n} [b(x) - b(y)] \frac{\Omega(x-y)}{|x-y|^{n-\alpha}} f(y) dy,$$

where $0 < \alpha < n$, and f is a suitable function.

Remark 1. [55] When Ω satisfies the specified size conditions, the kernel of the operator $\overline{T}_{\Omega,\alpha}$ has no regularity, so the operator $\overline{T}_{\Omega,\alpha}$ is called a rough fractional integral operator. In recent years, a variety of operators related to the fractional integrals, but lacking the smoothness required in the classical theory, have been studied. These include the operator $[b, \overline{T}_{\Omega,\alpha}]$. For more results, we refer the reader to [8, 17, 18, 22, 27, 28, 29, 30, 68].

In this paper, extending the definition of vanishing Morrey spaces [64] and vanishing generalized Morrey spaces [54], the author introduces the generalized vanishing local Morrey spaces $VL M_{p,\varphi}^{\{x_0\}}$, including their weak versions and studies the boundedness of the sublinear operators with rough kernel generated by fractional integrals and commutators in these spaces. These conditions are satisfied by most of the operators in harmonic analysis, such as fractional maximal operator, fractional integral operator(Riesz potential), fractional Marcinkiewicz operator and so on. In all the cases the conditions for the boundedness of $T_{\Omega,\alpha}$ and $T_{\Omega,b,\alpha}$ are given in terms of Zygmund-type integral inequalities on (φ_1, φ_2) , where there is no assumption on monotonicity of φ_1, φ_2 in r . As an example to the conditions of these theorems are satisfied, we can consider the Marcinkiewicz operator.

By $A \lesssim B$ we mean that $A \leq CB$ with some positive constant C independent of appropriate quantities. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$ and say that A and B are equivalent.

2. GENERALIZED VANISHING LOCAL MORREY SPACES

After studying Morrey spaces in detail, researchers have passed to generalized Morrey spaces. Mizuhara [42] has given generalized Morrey spaces $M_{p,\varphi}$ considering $\varphi = \varphi(r)$ instead of r^λ in the above definition of the Morrey space. Later, Guliyev [26] has defined the generalized Morrey spaces $M_{p,\varphi}$ with normalized norm as follows:

Definition 1. [26] (*generalized Morrey space*) Let $\varphi(x, r)$ be a positive measurable function on $\mathbb{R}^n \times (0, \infty)$ and $1 \leq p < \infty$. We denote by $M_{p,\varphi} \equiv M_{p,\varphi}(\mathbb{R}^n)$ the generalized Morrey space, the space of all functions $f \in L_p^{loc}(\mathbb{R}^n)$ with finite quasinorm

$$\|f\|_{M_{p,\varphi}} = \sup_{x \in \mathbb{R}^n, r > 0} \varphi(x, r)^{-1} |B(x, r)|^{-\frac{1}{p}} \|f\|_{L_p(B(x, r))}.$$

Also by $WM_{p,\varphi} \equiv WM_{p,\varphi}(\mathbb{R}^n)$ we denote the weak generalized Morrey space of all functions $f \in WL_p^{loc}(\mathbb{R}^n)$ for which

$$\|f\|_{WM_{p,\varphi}} = \sup_{x \in \mathbb{R}^n, r > 0} \varphi(x, r)^{-1} |B(x, r)|^{-\frac{1}{p}} \|f\|_{WL_p(B(x, r))} < \infty.$$

According to this definition, we recover the Morrey space $L_{p,\lambda}$ and weak Morrey space $WL_{p,\lambda}$ under the choice $\varphi(x, r) = r^{\frac{\lambda-n}{p}}$:

$$L_{p,\lambda} = M_{p,\varphi} \big|_{\varphi(x,r)=r^{\frac{\lambda-n}{p}}}, \quad WL_{p,\lambda} = WM_{p,\varphi} \big|_{\varphi(x,r)=r^{\frac{\lambda-n}{p}}}.$$

During the last decades various classical operators, such as maximal, singular and potential operators have been widely investigated in generalized Morrey spaces (see [16, 26, 28, 32, 56] for details).

Recall that in 2015 the work [5] and the Ph.D. thesis [28] by Gurbuz et al. have been introduced the generalized local Morrey space $LM_{p,\varphi}^{\{x_0\}}$ given by

Definition 2. (*generalized local Morrey space*) Let $\varphi(x, r)$ be a positive measurable function on $\mathbb{R}^n \times (0, \infty)$ and $1 \leq p < \infty$. For any fixed $x_0 \in \mathbb{R}^n$ we denote by $LM_{p,\varphi}^{\{x_0\}} \equiv LM_{p,\varphi}^{\{x_0\}}(\mathbb{R}^n)$ the generalized local Morrey space, the space of all functions $f \in L_p^{loc}(\mathbb{R}^n)$ with finite quasinorm

$$\|f\|_{LM_{p,\varphi}^{\{x_0\}}} = \sup_{r > 0} \varphi(x_0, r)^{-1} |B(x_0, r)|^{-\frac{1}{p}} \|f\|_{L_p(B(x_0, r))} < \infty.$$

Also by $WLM_{p,\varphi}^{\{x_0\}} \equiv WLM_{p,\varphi}^{\{x_0\}}(\mathbb{R}^n)$ we denote the weak generalized local Morrey space of all functions $f \in WL_p^{loc}(\mathbb{R}^n)$ for which

$$\|f\|_{WLM_{p,\varphi}^{\{x_0\}}} = \sup_{r > 0} \varphi(x_0, r)^{-1} |B(x_0, r)|^{-\frac{1}{p}} \|f\|_{WL_p(B(x_0, r))} < \infty.$$

According to this definition, we recover the local Morrey space $LM_{p,\lambda}^{\{x_0\}}$ and the weak local Morrey space $WLM_{p,\lambda}^{\{x_0\}}$ under the choice $\varphi(x_0, r) = r^{\frac{\lambda-n}{p}}$:

$$LL_{p,\lambda}^{\{x_0\}} = LM_{p,\varphi}^{\{x_0\}} \big|_{\varphi(x_0,r)=r^{\frac{\lambda-n}{p}}}, \quad WLL_{p,\lambda}^{\{x_0\}} = WLM_{p,\varphi}^{\{x_0\}} \big|_{\varphi(x_0,r)=r^{\frac{\lambda-n}{p}}}.$$

The main goal of [5] and [28] is to give some sufficient conditions for the boundedness of a large class of rough sublinear operators and their commutators on the generalized local Morrey space $LM_{p,\varphi}^{\{x_0\}}$. For the properties and applications of generalized local Morrey spaces $LM_{p,\varphi}^{\{x_0\}}$, see also [28].

Furthermore, we have the following embeddings:

$$M_{p,\varphi} \subset LM_{p,\varphi}^{\{x_0\}}, \quad \|f\|_{LM_{p,\varphi}^{\{x_0\}}} \leq \|f\|_{M_{p,\varphi}},$$

$$WM_{p,\varphi} \subset WLM_{p,\varphi}^{\{x_0\}}, \quad \|f\|_{WLM_{p,\varphi}^{\{x_0\}}} \leq \|f\|_{WM_{p,\varphi}}.$$

Wiener [66, 67] has looked for a way to describe the behavior of a function at the infinity. The conditions he considered are related to appropriate weighted L_q spaces. Beurling [4] has extended this idea and has defined a pair of dual Banach spaces A_q and $B_{q'}$, where $1/q + 1/q' = 1$. To be precise, A_q is a Banach algebra with respect to the convolution, expressed as a union of certain weighted L_q spaces; the space $B_{q'}$ is expressed as the intersection of the corresponding weighted $L_{q'}$ spaces. Feichtinger [21] has observed that the space B_q can be described by

$$(2.1) \quad \|f\|_{B_q} = \sup_{k \geq 0} 2^{-\frac{kn}{q}} \|f\chi_k\|_{L_q(\mathbb{R}^n)} < \infty,$$

where χ_0 is the characteristic function of the unit ball $\{x \in \mathbb{R}^n : |x| \leq 1\}$, χ_k is the characteristic function of the annulus $\{x \in \mathbb{R}^n : 2^{k-1} < |x| \leq 2^k\}$, $k = 1, 2, \dots$. By duality, the space $A_q(\mathbb{R}^n)$, appropriately called now the Beurling algebra, can be described by the condition

$$(2.2) \quad \|f\|_{A_q} = \sum_{k=0}^{\infty} 2^{\frac{kn}{q'}} \|f\chi_k\|_{L_q(\mathbb{R}^n)} < \infty.$$

Let $\dot{B}_q(\mathbb{R}^n)$ and $\dot{A}_q(\mathbb{R}^n)$ be the homogeneous versions of $B_q(\mathbb{R}^n)$ and $A_q(\mathbb{R}^n)$ by taking $k \in \mathbb{Z}$ in (2.1) and (2.2) instead of $k \geq 0$ there.

If $\lambda < 0$ or $\lambda > n$, then $LM_{p,\lambda}^{\{x_0\}}(\mathbb{R}^n) = \Theta$, where Θ is the set of all functions equivalent to 0 on \mathbb{R}^n . Note that $LM_{p,0}(\mathbb{R}^n) = L_p(\mathbb{R}^n)$ and $LM_{p,n}(\mathbb{R}^n) = \dot{B}_p(\mathbb{R}^n)$.

$$\dot{B}_{p,\mu} = LM_{p,\varphi}|_{\varphi(0,r)=r^{\mu n}}, \quad W\dot{B}_{p,\mu} = WLM_{p,\varphi}|_{\varphi(0,r)=r^{\mu n}}.$$

Alvarez et al. [3], in order to study the relationship between central BMO spaces and Morrey spaces, they introduced λ -central bounded mean oscillation spaces and central Morrey spaces $\dot{B}_{p,\mu}(\mathbb{R}^n) \equiv LM_{p,n+np\mu}(\mathbb{R}^n)$, $\mu \in [-\frac{1}{p}, 0]$. If $\mu < -\frac{1}{p}$ or $\mu > 0$, then $\dot{B}_{p,\mu}(\mathbb{R}^n) = \Theta$. Note that $\dot{B}_{p,-\frac{1}{p}}(\mathbb{R}^n) = L_p(\mathbb{R}^n)$ and $\dot{B}_{p,0}(\mathbb{R}^n) = \dot{B}_p(\mathbb{R}^n)$. Also define the weak central Morrey spaces $W\dot{B}_{p,\mu}(\mathbb{R}^n) \equiv WLM_{p,n+np\mu}(\mathbb{R}^n)$.

The vanishing generalized Morrey spaces $VM_{p,\varphi}(\mathbb{R}^n)$ which has been introduced and studied by Samko [54] is defined as follows.

Definition 3. (vanishing generalized Morrey space) Let $\varphi(x, r)$ be a positive measurable function on $\mathbb{R}^n \times (0, \infty)$ and $1 \leq p < \infty$. The vanishing generalized Morrey space $VM_{p,\varphi}(\mathbb{R}^n)$ is defined as the spaces of functions $f \in L_p^{loc}(\mathbb{R}^n)$ such that

$$\lim_{r \rightarrow 0} \sup_{x \in \mathbb{R}^n} \varphi(x, r)^{-1} \int_{B(x, r)} |f(y)|^p dy = 0.$$

Everywhere in the sequel we assume that

$$\lim_{t \rightarrow 0} \frac{t^{\frac{n}{p}}}{\varphi(x, t)} = 0,$$

and

$$\sup_{0 < t < \infty} \frac{t^{\frac{n}{p}}}{\varphi(x, t)} < \infty,$$

which make the spaces $VM_{p,\varphi}(\mathbb{R}^n)$ non-trivial, because bounded functions with compact support belong to this space. The spaces $VM_{p,\varphi}(\mathbb{R}^n)$ and $WVM_{p,\varphi}(\mathbb{R}^n)$ are Banach spaces with respect to the norm (see, for example [54])

$$\|f\|_{VM_{p,\varphi}} = \sup_{x \in \mathbb{R}^n, r > 0} \varphi(x, r)^{-1} \|f\|_{L_p(B(x, r))},$$

$$\|f\|_{WVM_{p,\varphi}} = \sup_{x \in \mathbb{R}^n, r > 0} \varphi(x, r)^{-1} \|f\|_{WL_p(B(x, r))},$$

respectively. The spaces $VM_{p,\varphi}(\mathbb{R}^n)$ and $WVM_{p,\varphi}(\mathbb{R}^n)$ are closed subspaces of the Banach spaces $M_{p,\varphi}(\mathbb{R}^n)$ and $WM_{p,\varphi}(\mathbb{R}^n)$, respectively, which may be shown by standard means.

Furthermore, we have the following embeddings:

$$VM_{p,\varphi} \subset M_{p,\varphi}, \quad \|f\|_{M_{p,\varphi}} \leq \|f\|_{VM_{p,\varphi}},$$

$$WVM_{p,\varphi} \subset WM_{p,\varphi}, \quad \|f\|_{WM_{p,\varphi}} \leq \|f\|_{WVM_{p,\varphi}}.$$

For the properties and applications of vanishing generalized Morrey spaces, see also [2, 54].

For brevity, in the sequel we use the notations

$$\mathfrak{M}_{p,\varphi}(f; x_0, r) := \frac{|B(x_0, r)|^{-\frac{1}{p}} \|f\|_{L_p(B(x_0, r))}}{\varphi(x_0, r)}$$

and

$$\mathfrak{M}_{p,\varphi}^W(f; x_0, r) := \frac{|B(x_0, r)|^{-\frac{1}{p}} \|f\|_{WL_p(B(x_0, r))}}{\varphi(x_0, r)}.$$

Extending the definition of vanishing generalized Morrey spaces to the case of generalized local Morrey spaces, we introduce the following definitions.

Definition 4. (generalized vanishing local Morrey space) *The generalized vanishing local Morrey space $VLM_{p,\varphi}^{\{x_0\}}(\mathbb{R}^n)$ is defined as the spaces of functions $f \in LM_{p,\varphi}^{\{x_0\}}(\mathbb{R}^n)$ such that*

$$(2.3) \quad \lim_{r \rightarrow 0} \mathfrak{M}_{p,\varphi}(f; x_0, r) = 0.$$

Definition 5. (weak generalized vanishing local Morrey space) *The weak generalized vanishing local Morrey space $WVLM_{p,\varphi}^{\{x_0\}}(\mathbb{R}^n)$ is defined as the spaces of functions $f \in WLM_{p,\varphi}^{\{x_0\}}(\mathbb{R}^n)$ such that*

$$(2.4) \quad \lim_{r \rightarrow 0} \mathfrak{M}_{p,\varphi}^W(f; x_0, r) = 0.$$

Everywhere in the sequel we assume that

$$(2.5) \quad \lim_{r \rightarrow 0} \frac{1}{\varphi(x_0, r)} = 0,$$

and

$$(2.6) \quad \sup_{0 < r < \infty} \frac{1}{\varphi(x_0, r)} < \infty,$$

which make the spaces $VLM_{p,\varphi}^{\{x_0\}}(\mathbb{R}^n)$ non-trivial, because bounded functions with compact support belong to this space. The spaces $VLM_{p,\varphi}^{\{x_0\}}(\mathbb{R}^n)$ and $WVLM_{p,\varphi}^{\{x_0\}}(\mathbb{R}^n)$ are Banach spaces with respect to the norm

$$(2.7) \quad \|f\|_{VLM_{p,\varphi}^{\{x_0\}}} \equiv \|f\|_{LM_{p,\varphi}^{\{x_0\}}} = \sup_{r>0} \mathfrak{M}_{p,\varphi}(f; x_0, r),$$

$$(2.8) \quad \|f\|_{WVLM_{p,\varphi}^{\{x_0\}}} = \|f\|_{WLM_{p,\varphi}^{\{x_0\}}} = \sup_{r>0} \mathfrak{M}_{p,\varphi}^W(f; x_0, r),$$

respectively. The spaces $VLM_{p,\varphi}^{\{x_0\}}(\mathbb{R}^n)$ and $WVLM_{p,\varphi}^{\{x_0\}}(\mathbb{R}^n)$ are closed subspaces of the Banach spaces $LM_{p,\varphi}^{\{x_0\}}(\mathbb{R}^n)$ and $WLM_{p,\varphi}^{\{x_0\}}(\mathbb{R}^n)$, respectively, which may be shown by standard means.

3. SUBLINEAR OPERATORS WITH ROUGH KERNEL $T_{\Omega,\alpha}$ ON THE SPACES $LM_{p,\varphi}^{\{x_0\}}$ AND $VLM_{p,\varphi}^{\{x_0\}}$

In this section, we will first prove the boundedness of the operator $T_{\Omega,\alpha}$ satisfying (1.1) on the generalized local Morrey spaces $LM_{p,\varphi}^{\{x_0\}}(\mathbb{R}^n)$, including their weak versions by using the following statement on the boundedness of the weighted Hardy operator

$$H_\omega g(t) := \int_t^\infty g(s)\omega(s)ds, \quad 0 < t < \infty,$$

where ω is a fixed non-negative function and measurable on $(0, \infty)$. Then, We will also give the boundedness of $T_{\Omega,\alpha}$ satisfying (1.1) on generalized vanishing local Morrey spaces $VLM_{p,\varphi}^{\{x_0\}}(\mathbb{R}^n)$, including their weak versions.

Theorem 3. [5, 27, 28, 29, 30] *Let v_1, v_2 and ω be positive almost everywhere and measurable functions on $(0, \infty)$. The inequality*

$$(3.1) \quad \operatorname{esssup}_{t>0} v_2(t) H_\omega g(t) \leq C \operatorname{esssup}_{t>0} v_1(t) g(t)$$

holds for some $C > 0$ for all non-negative and non-decreasing functions g on $(0, \infty)$ if and only if

$$(3.2) \quad B := \sup_{t>0} v_2(t) \int_t^\infty \frac{\omega(s)ds}{\operatorname{esssup}_{s<\tau<\infty} v_1(\tau)} < \infty.$$

Moreover, the value $C = B$ is the best constant for (3.1).

We first prove the following Lemma 2.

Lemma 2. *Suppose that $x_0 \in \mathbb{R}^n$, $\Omega \in L_s(S^{n-1})$, $1 < s \leq \infty$, is homogeneous of degree zero. Let $0 < \alpha < n$, $1 \leq p < \frac{n}{\alpha}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. Let $T_{\Omega,\alpha}$ be a sublinear operator satisfying condition (1.1), bounded from $L_p(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$ for $p > 1$, and bounded from $L_1(\mathbb{R}^n)$ to $WL_q(\mathbb{R}^n)$ for $p = 1$.*

If $p > 1$ and $s' \leq p$, then the inequality

$$(3.3) \quad \|T_{\Omega,\alpha} f\|_{L_q(B(x_0,r))} \lesssim r^{\frac{n}{q}} \int_{2r}^\infty t^{-\frac{n}{q}-1} \|f\|_{L_p(B(x_0,t))} dt$$

holds for any ball $B(x_0, r)$ and for all $f \in L_p^{loc}(\mathbb{R}^n)$.

If $p > 1$ and $q < s$, then the inequality

$$\|T_{\Omega,\alpha}f\|_{L_q(B(x_0,r))} \lesssim r^{\frac{n}{q}-\frac{n}{s}} \int_{2r}^{\infty} t^{\frac{n}{s}-\frac{n}{q}-1} \|f\|_{L_p(B(x_0,t))} dt$$

holds for any ball $B(x_0, r)$ and for all $f \in L_p^{loc}(\mathbb{R}^n)$.

Moreover, for $p = 1 < q < s$ the inequality

$$(3.4) \quad \|T_{\Omega,\alpha}f\|_{WL_q(B(x_0,r))} \lesssim r^{\frac{n}{q}} \int_{2r}^{\infty} t^{-\frac{n}{q}-1} \|f\|_{L_1(B(x_0,t))} dt$$

holds for any ball $B(x_0, r)$ and for all $f \in L_1^{loc}(\mathbb{R}^n)$.

Proof. Let $0 < \alpha < n$, $1 \leq s' < p < \frac{n}{\alpha}$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. Set $B = B(x_0, r)$ for the ball centered at x_0 and of radius r and $2B = B(x_0, 2r)$. We represent f as

$$(3.5) \quad f = f_1 + f_2, \quad f_1(y) = f(y) \chi_{2B}(y), \quad f_2(y) = f(y) \chi_{(2B)^c}(y), \quad r > 0$$

and have

$$\|T_{\Omega,\alpha}f\|_{L_q(B)} \leq \|T_{\Omega,\alpha}f_1\|_{L_q(B)} + \|T_{\Omega,\alpha}f_2\|_{L_q(B)}.$$

Since $f_1 \in L_p(\mathbb{R}^n)$, $T_{\Omega,\alpha}f_1 \in L_q(\mathbb{R}^n)$ and from the boundedness of $T_{\Omega,\alpha}$ from $L_p(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$ (see Lemma 1) it follows that:

$$\|T_{\Omega,\alpha}f_1\|_{L_q(B)} \leq \|T_{\Omega,\alpha}f_1\|_{L_q(\mathbb{R}^n)} \leq C \|f_1\|_{L_p(\mathbb{R}^n)} = C \|f\|_{L_p(2B)},$$

where constant $C > 0$ is independent of f .

It is clear that $x \in B$, $y \in (2B)^c$ implies $\frac{1}{2}|x_0 - y| \leq |x - y| \leq \frac{3}{2}|x_0 - y|$. We get

$$|T_{\Omega,\alpha}f_2(x)| \leq 2^{n-\alpha} c_1 \int_{(2B)^c} \frac{|f(y)| |\Omega(x-y)|}{|x_0 - y|^{n-\alpha}} dy.$$

By the Fubini's theorem, we have

$$\begin{aligned} \int_{(2B)^c} \frac{|f(y)| |\Omega(x-y)|}{|x_0 - y|^{n-\alpha}} dy &\approx \int_{(2B)^c} |f(y)| |\Omega(x-y)| \int_{|x_0-y|}^{\infty} \frac{dt}{t^{n+1-\alpha}} dy \\ &\approx \int_{2r}^{\infty} \int_{2r \leq |x_0-y| \leq t} |f(y)| |\Omega(x-y)| dy \frac{dt}{t^{n+1-\alpha}} \\ &\lesssim \int_{2r}^{\infty} \int_{B(x_0,t)} |f(y)| |\Omega(x-y)| dy \frac{dt}{t^{n+1-\alpha}}. \end{aligned}$$

Applying the Hölder's inequality, we get

$$\begin{aligned} &\int_{(2B)^c} \frac{|f(y)| |\Omega(x-y)|}{|x_0 - y|^{n-\alpha}} dy \\ (3.6) \quad &\lesssim \int_{2r}^{\infty} \|f\|_{L_p(B(x_0,t))} \|\Omega(x-\cdot)\|_{L_s(B(x_0,t))} |B(x_0,t)|^{1-\frac{1}{p}-\frac{1}{s}} \frac{dt}{t^{n+1-\alpha}}. \end{aligned}$$

For $x \in B(x_0, t)$, notice that Ω is homogenous of degree zero and $\Omega \in L_s(S^{n-1})$, $s > 1$. Then, we obtain

$$\begin{aligned}
 \left(\int_{B(x_0, t)} |\Omega(x-y)|^s dy \right)^{\frac{1}{s}} &= \left(\int_{B(x-x_0, t)} |\Omega(z)|^s dz \right)^{\frac{1}{s}} \\
 &\leq \left(\int_{B(0, t+|x-x_0|)} |\Omega(z)|^s dz \right)^{\frac{1}{s}} \\
 &\leq \left(\int_{B(0, 2t)} |\Omega(z)|^s dz \right)^{\frac{1}{s}} \\
 &= \left(\int_{S^{n-1}} \int_0^{2t} |\Omega(z')|^s d\sigma(z') r^{n-1} dr \right)^{\frac{1}{s}} \\
 (3.7) \quad &= C \|\Omega\|_{L_s(S^{n-1})} |B(x_0, 2t)|^{\frac{1}{s}}.
 \end{aligned}$$

Thus, by (3.7), it follows that:

$$|T_{\Omega, \alpha} f_2(x)| \lesssim \int_{2r}^{\infty} \|f\|_{L_p(B(x_0, t))} \frac{dt}{t^{\frac{n}{q}+1}}.$$

Moreover, for all $p \in [1, \infty)$ the inequality

$$(3.8) \quad \|T_{\Omega, \alpha} f_2\|_{L_q(B)} \lesssim r^{\frac{n}{q}} \int_{2r}^{\infty} \|f\|_{L_p(B(x_0, t))} \frac{dt}{t^{\frac{n}{q}+1}}$$

is valid. Thus, we obtain

$$\|T_{\Omega, \alpha} f\|_{L_q(B)} \lesssim \|f\|_{L_p(2B)} + r^{\frac{n}{q}} \int_{2r}^{\infty} \|f\|_{L_p(B(x_0, t))} \frac{dt}{t^{\frac{n}{q}+1}}.$$

On the other hand, we have

$$\begin{aligned}
 \|f\|_{L_p(2B)} &\approx r^{\frac{n}{q}} \|f\|_{L_p(2B)} \int_{2r}^{\infty} \frac{dt}{t^{\frac{n}{q}+1}} \\
 (3.9) \quad &\leq r^{\frac{n}{q}} \int_{2r}^{\infty} \|f\|_{L_p(B(x_0, t))} \frac{dt}{t^{\frac{n}{q}+1}}.
 \end{aligned}$$

By combining the above inequalities, we obtain

$$\|T_{\Omega, \alpha} f\|_{L_q(B)} \lesssim r^{\frac{n}{q}} \int_{2r}^{\infty} \|f\|_{L_p(B(x_0, t))} \frac{dt}{t^{\frac{n}{q}+1}}.$$

Let $1 < q < s$. Similarly to (3.7), when $y \in B(x_0, t)$, it is true that

$$(3.10) \quad \left(\int_{B(x_0, r)} |\Omega(x-y)|^s dy \right)^{\frac{1}{s}} \leq C \|\Omega\|_{L_s(S^{n-1})} \left| B\left(x_0, \frac{3}{2}t\right) \right|^{\frac{1}{s}}.$$

By the Fubini's theorem, the Minkowski inequality and (3.10), we get

$$\begin{aligned} \|T_{\Omega, \alpha} f_2\|_{L_q(B)} &\leq \left(\int_B \left| \int_{2r}^{\infty} \int_{B(x_0, t)} |f(y)| |\Omega(x-y)| dy \frac{dt}{t^{n+1-\alpha}} \right|^q dx \right)^{\frac{1}{q}} \\ &\leq \int_{2r}^{\infty} \int_{B(x_0, t)} |f(y)| \|\Omega(\cdot - y)\|_{L_q(B)} dy \frac{dt}{t^{n+1-\alpha}} \\ &\leq |B(x_0, r)|^{\frac{1}{q} - \frac{1}{s}} \int_{2r}^{\infty} \int_{B(x_0, t)} |f(y)| \|\Omega(\cdot - y)\|_{L_s(B)} dy \frac{dt}{t^{n+1-\alpha}} \\ &\lesssim r^{\frac{n}{q} - \frac{n}{s}} \int_{2r}^{\infty} \|f\|_{L_1(B(x_0, t))} \left| B\left(x_0, \frac{3}{2}t\right) \right|^{\frac{1}{s}} \frac{dt}{t^{n+1-\alpha}} \\ &\lesssim r^{\frac{n}{q} - \frac{n}{s}} \int_{2r}^{\infty} \|f\|_{L_p(B(x_0, t))} t^{\frac{n}{s} - \frac{n}{q} - 1} dt. \end{aligned}$$

Let $p = 1 < q < s \leq \infty$. From the weak $(1, q)$ boundedness of $T_{\Omega, \alpha}$ and (3.9) it follows that:

$$(3.11) \quad \begin{aligned} \|T_{\Omega, \alpha} f_1\|_{WL_q(B)} &\leq \|T_{\Omega, \alpha} f_1\|_{WL_q(\mathbb{R}^n)} \lesssim \|f_1\|_{L_1(\mathbb{R}^n)} \\ &= \|f\|_{L_1(2B)} \lesssim r^{\frac{n}{q}} \int_{2r}^{\infty} \|f\|_{L_1(B(x_0, t))} \frac{dt}{t^{\frac{n}{q}+1}}. \end{aligned}$$

Then from (3.8) and (3.11) we get the inequality (3.4), which completes the proof. \square

In the following theorem (our main result), we get the boundedness of the operator $T_{\Omega, \alpha}$ on the generalized local Morrey spaces $LM_{p, \varphi}^{\{x_0\}}$.

Theorem 4. *Suppose that $x_0 \in \mathbb{R}^n$, $\Omega \in L_s(S^{n-1})$, $1 < s \leq \infty$, is homogeneous of degree zero. Let $0 < \alpha < n$, $1 \leq p < \frac{n}{\alpha}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. Let $T_{\Omega, \alpha}$ be a sublinear operator satisfying condition (1.1), bounded from $L_p(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$ for $p > 1$, and bounded from $L_1(\mathbb{R}^n)$ to $WL_q(\mathbb{R}^n)$ for $p = 1$. Let also, for $s' \leq p$, $p \neq 1$, the pair (φ_1, φ_2) satisfies the condition*

$$(3.12) \quad \int_r^{\infty} \frac{\text{essinf}_{t < \tau < \infty} \varphi_1(x_0, \tau) \tau^{\frac{n}{p}}}{t^{\frac{n}{q}+1}} dt \leq C \varphi_2(x_0, r),$$

and for $q < s$ the pair (φ_1, φ_2) satisfies the condition

$$(3.13) \quad \int_r^\infty \frac{\operatorname{ess\,inf}_{t < \tau < \infty} \varphi_1(x_0, \tau) \tau^{\frac{n}{p}}}{t^{\frac{n}{q} - \frac{n}{s} + 1}} dt \leq C \varphi_2(x_0, r) r^{\frac{n}{s}},$$

where C does not depend on r .

Then the operator $T_{\Omega, \alpha}$ is bounded from $LM_{p, \varphi_1}^{\{x_0\}}$ to $LM_{q, \varphi_2}^{\{x_0\}}$ for $p > 1$ and from $LM_{1, \varphi_1}^{\{x_0\}}$ to $WLM_{q, \varphi_2}^{\{x_0\}}$ for $p = 1$. Moreover, we have for $p > 1$

$$(3.14) \quad \|T_{\Omega, \alpha} f\|_{LM_{q, \varphi_2}^{\{x_0\}}} \lesssim \|f\|_{LM_{p, \varphi_1}^{\{x_0\}}},$$

and for $p = 1$

$$(3.15) \quad \|T_{\Omega, \alpha} f\|_{WLM_{q, \varphi_2}^{\{x_0\}}} \lesssim \|f\|_{LM_{1, \varphi_1}^{\{x_0\}}}.$$

Proof. Let $1 < p < \infty$ and $s' \leq p$. By Lemma 2 and Theorem 3 with $v_2(r) = \varphi_2(x_0, r)^{-1}$, $v_1 = \varphi_1(x_0, r)^{-1} r^{-\frac{n}{p}}$, $w(r) = r^{-\frac{n}{q} - 1}$ and $g(r) = \|f\|_{L_p(B(x_0, r))}$, we have

$$\begin{aligned} \|T_{\Omega, \alpha} f\|_{LM_{q, \varphi_2}^{\{x_0\}}} &\lesssim \sup_{r>0} \varphi_2(x_0, r)^{-1} \int_r^\infty \|f\|_{L_p(B(x_0, t))} \frac{dt}{t^{\frac{n}{q} + 1}} \\ &\lesssim \sup_{r>0} \varphi_1(x_0, r)^{-1} r^{-\frac{n}{p}} \|f\|_{L_p(B(x_0, r))} = \|f\|_{LM_{p, \varphi_1}^{\{x_0\}}}, \end{aligned}$$

where condition (3.2) is equivalent to (3.12), then we obtain (3.14).

Let $1 \leq q < s$. By Lemma 2 and Theorem 3 with $v_2(r) = \varphi_2(x_0, r)^{-1}$, $v_1 = \varphi_1(x_0, r)^{-1} r^{-\frac{n}{p} + \frac{n}{s}}$, $w(r) = r^{-\frac{n}{q} + \frac{n}{s} - 1}$ and $g(r) = \|f\|_{L_p(B(x_0, r))}$, we have

$$\begin{aligned} \|T_{\Omega, \alpha} f\|_{LM_{q, \varphi_2}^{\{x_0\}}} &\lesssim \sup_{r>0} \varphi_2(x_0, r)^{-1} r^{-\frac{n}{s}} \int_r^\infty \|f\|_{L_p(B(x_0, t))} \frac{dt}{t^{\frac{n}{q} - \frac{n}{s} + 1}} \\ &\lesssim \sup_{r>0} \varphi_1(x_0, r)^{-1} r^{-\frac{n}{p}} \|f\|_{L_p(B(x_0, r))} = \|f\|_{LM_{p, \varphi_1}^{\{x_0\}}}, \end{aligned}$$

where condition (3.2) is equivalent to (3.13). Thus, we obtain (3.14).

Also, for $p = 1$ we have

$$\begin{aligned} \|T_{\Omega, \alpha} f\|_{WLM_{q, \varphi_2}^{\{x_0\}}} &\lesssim \sup_{r>0} \varphi_2(x_0, r)^{-1} \int_r^\infty \|f\|_{L_1(B(x_0, t))} \frac{dt}{t^{\frac{n}{q} + 1}} \\ &\lesssim \sup_{r>0} \varphi_1(x_0, r)^{-1} r^{-n} \|f\|_{L_1(B(x_0, r))} = \|f\|_{LM_{1, \varphi_1}^{\{x_0\}}}. \end{aligned}$$

Hence, the proof is completed. \square

In the case of $q = \infty$ by Theorem 4, we get

Corollary 2. Let $x_0 \in \mathbb{R}^n$, $1 \leq p < \infty$, $0 < \alpha < \frac{n}{p}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$ and the pair (φ_1, φ_2) satisfies condition (3.12). Then the operators M_α and \bar{T}_α are bounded from $LM_{p, \varphi_1}^{\{x_0\}}$ to $LM_{q, \varphi_2}^{\{x_0\}}$ for $p > 1$ and from $LM_{1, \varphi_1}^{\{x_0\}}$ to $WLM_{q, \varphi_2}^{\{x_0\}}$ for $p = 1$.

Corollary 3. Suppose that $x_0 \in \mathbb{R}^n$, $\Omega \in L_s(S^{n-1})$, $1 < s \leq \infty$, is homogeneous of degree zero. Let $0 < \alpha < n$, $1 \leq p < \frac{n}{\alpha}$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. Let also for $s' \leq p$ the pair (φ_1, φ_2) satisfies condition (3.12) and for $q < s$ the pair (φ_1, φ_2) satisfies

condition (3.13). Then the operators $M_{\Omega,\alpha}$ and $\overline{T}_{\Omega,\alpha}$ are bounded from $LM_{p,\varphi_1}^{\{x_0\}}$ to $LM_{q,\varphi_2}^{\{x_0\}}$ for $p > 1$ and from $LM_{1,\varphi_1}^{\{x_0\}}$ to $WLM_{q,\varphi_2}^{\{x_0\}}$ for $p = 1$.

Now using above results, we get the boundedness of the operator $T_{\Omega,\alpha}$ on the generalized vanishing local Morrey spaces $VLM_{p,\varphi}^{\{x_0\}}$.

Theorem 5. (Our main result) Let $x_0 \in \mathbb{R}^n$, $\Omega \in L_s(S^{n-1})$, $1 < s \leq \infty$, be homogeneous of degree zero. Let $0 < \alpha < n$, $1 \leq p < \frac{n}{\alpha}$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. Let $T_{\Omega,\alpha}$ be a sublinear operator satisfying condition (1.1), bounded on $L_p(\mathbb{R}^n)$ for $p > 1$, and bounded from $L_1(\mathbb{R}^n)$ to $WL_1(\mathbb{R}^n)$. Let for $s' \leq p$, $p \neq 1$, the pair (φ_1, φ_2) satisfies conditions (2.5)-(2.6) and

$$(3.16) \quad c_\delta := \int_{\delta}^{\infty} \varphi_1(x_0, t) \frac{t^{\frac{n}{p}}}{t^{\frac{n}{q}+1}} dt < \infty$$

for every $\delta > 0$, and

$$(3.17) \quad \int_r^{\infty} \varphi_1(x_0, t) \frac{t^{\frac{n}{p}}}{t^{\frac{n}{q}+1}} dt \leq C_0 \varphi_2(x_0, r),$$

and for $q < s$ the pair (φ_1, φ_2) satisfies conditions (2.5)-(2.6) and also

$$(3.18) \quad c_{\delta'} := \int_{\delta'}^{\infty} \varphi_1(x_0, t) \frac{t^{\frac{n}{p}}}{t^{\frac{n}{q}-\frac{n}{s}+1}} dt < \infty$$

for every $\delta' > 0$, and

$$(3.19) \quad \int_r^{\infty} \varphi_1(x_0, t) \frac{t^{\frac{n}{p}}}{t^{\frac{n}{q}-\frac{n}{s}+1}} dt \leq C_0 \varphi_2(x_0, r) r^{\frac{n}{s}},$$

where C_0 does not depend on $r > 0$.

Then the operator $T_{\Omega,\alpha}$ is bounded from $VLM_{p,\varphi_1}^{\{x_0\}}$ to $VLM_{q,\varphi_2}^{\{x_0\}}$ for $p > 1$ and from $VLM_{1,\varphi_1}^{\{x_0\}}$ to $WVLM_{q,\varphi_2}^{\{x_0\}}$ for $p = 1$. Moreover, we have for $p > 1$

$$(3.20) \quad \|T_{\Omega,\alpha} f\|_{VLM_{q,\varphi_2}^{\{x_0\}}} \lesssim \|f\|_{VLM_{p,\varphi_1}^{\{x_0\}}},$$

and for $p = 1$

$$(3.21) \quad \|T_{\Omega,\alpha} f\|_{WVLM_{q,\varphi_2}^{\{x_0\}}} \lesssim \|f\|_{VLM_{1,\varphi_1}^{\{x_0\}}}.$$

Proof. The norm inequalities follow from Theorem 4. Thus we only have to prove that

$$(3.22) \quad \lim_{r \rightarrow 0} \mathfrak{M}_{p,\varphi_1}(f; x_0, r) = 0 \text{ implies } \lim_{r \rightarrow 0} \mathfrak{M}_{q,\varphi_2}(T_{\Omega,\alpha} f; x_0, r) = 0$$

and

$$(3.23) \quad \lim_{r \rightarrow 0} \mathfrak{M}_{p,\varphi_1}(f; x_0, r) = 0 \text{ implies } \lim_{r \rightarrow 0} \mathfrak{M}_{q,\varphi_2}^W(T_{\Omega,\alpha} f; x_0, r) = 0.$$

To show that $\frac{r^{-\frac{n}{q}} \|T_{\Omega,\alpha} f\|_{L_q(B(x_0, r))}}{\varphi_2(x_0, r)} < \epsilon$ for small r , we split the right-hand side of (3.3):

$$(3.24) \quad \frac{r^{-\frac{n}{q}} \|T_{\Omega,\alpha} f\|_{L_q(B(x_0, r))}}{\varphi_2(x_0, r)} \leq C [I_{\delta_0}(x_0, r) + J_{\delta_0}(x_0, r)],$$

where $\delta_0 > 0$ (we may take $\delta_0 < 1$), and

$$I_{\delta_0}(x_0, r) := \frac{1}{\varphi_2(x_0, r)} \int_r^{\delta_0} t^{-\frac{n}{q}-1} \|f\|_{L_p(B(x_0, t))} dt,$$

and

$$J_{\delta_0}(x_0, r) := \frac{1}{\varphi_2(x_0, r)} \int_{\delta_0}^{\infty} t^{-\frac{n}{q}-1} \|f\|_{L_p(B(x_0, t))} dt,$$

and $r < \delta_0$. Now we use the fact that $f \in VLM_{p, \varphi_1}^{\{x_0\}}$ and we choose any fixed $\delta_0 > 0$ such that

$$\frac{t^{-\frac{n}{p}} \|f\|_{L_p(B(x_0, t))}}{\varphi_1(x_0, t)} < \frac{\epsilon}{2CC_0}, \quad t \leq \delta_0,$$

where C and C_0 are constants from (3.17) and (3.24). This allows to estimate the first term uniformly in $r \in (0, \delta_0)$:

$$CI_{\delta_0}(x_0, r) < \frac{\epsilon}{2}, \quad 0 < r < \delta_0.$$

The estimation of the second term may be obtained by choosing r sufficiently small. Indeed, we have

$$J_{\delta_0}(x_0, r) \leq c_{\delta_0} \frac{\|f\|_{LM_{p, \varphi_1}^{\{x_0\}}}}{\varphi_2(x_0, r)},$$

where c_{δ_0} is the constant from (3.16) with $\delta = \delta_0$. Then, by (2.5) it suffices to choose r small enough such that

$$\frac{1}{\varphi_2(x_0, r)} \leq \frac{\epsilon}{2c_{\delta_0} \|f\|_{LM_{p, \varphi_1}^{\{x_0\}}}},$$

which completes the proof of (3.22).

The proof of (3.23) is similar to the proof of (3.22). For the case of $q < s$, we can also use the same method, so we omit the details. \square

Remark 2. Conditions (3.16) and (3.18) are not needed in the case when $\varphi(x, r)$ does not depend on x , since (3.16) follows from (3.17) and similarly, (3.18) follows from (3.19) in this case.

Corollary 4. Let $x_0 \in \mathbb{R}^n$, $\Omega \in L_s(S^{n-1})$, $1 < s \leq \infty$, be homogeneous of degree zero. Let $0 < \alpha < n$, $1 \leq p < \frac{n}{\alpha}$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. Let also for $s' \leq p$, $p \neq 1$, the pair (φ_1, φ_2) satisfies conditions (2.5)-(2.6) and (3.16)-(3.17) and for $q < s$ the pair (φ_1, φ_2) satisfies conditions (2.5)-(2.6) and (3.18)-(3.19). Then the operators $M_{\Omega, \alpha}$ and $\overline{T}_{\Omega, \alpha}$ are bounded from $VLM_{p, \varphi_1}^{\{x_0\}}$ to $VLM_{q, \varphi_2}^{\{x_0\}}$ for $p > 1$ and from $VLM_{1, \varphi_1}^{\{x_0\}}$ to $WVLM_{q, \varphi_2}^{\{x_0\}}$ for $p = 1$.

In the case of $q = \infty$ by Theorem 5, we get

Corollary 5. Let $x_0 \in \mathbb{R}^n$, $1 \leq p < \infty$ and the pair (φ_1, φ_2) satisfies conditions (2.5)-(2.6) and (3.16)-(3.17). Then the operators M_{α} and \overline{T}_{α} are bounded from $VLM_{p, \varphi_1}^{\{x_0\}}$ to $VLM_{q, \varphi_2}^{\{x_0\}}$ for $p > 1$ and from $VLM_{1, \varphi_1}^{\{x_0\}}$ to $WVLM_{q, \varphi_2}^{\{x_0\}}$ for $p = 1$.

4. COMMUTATORS OF THE LINEAR OPERATORS WITH ROUGH KERNEL $T_{\Omega,\alpha}$ ON THE SPACES $LM_{p,\varphi}^{\{x_0\}}$ AND $VL M_{p,\varphi}^{\{x_0\}}$

In this section, we will first prove the boundedness of the operator $T_{\Omega,b,\alpha}$ with $b \in LC_{p_2,\lambda}^{\{x_0\}}$ on the generalized local Morrey spaces $LM_{p,\varphi}^{\{x_0\}}$ by using the following weighted Hardy operator

$$H_\omega^* g(r) := \int_r^\infty \left(1 + \ln \frac{t}{r}\right) g(t) \omega(t) dt, \quad r \in (0, \infty),$$

where ω is a weight function. Then, we will also obtain the boundedness of $T_{\Omega,b,\alpha}$ with $b \in LC_{p_2,\lambda}^{\{x_0\}}$ on generalized vanishing local Morrey spaces $VL M_{p,\varphi}^{\{x_0\}}$.

Let T be a linear operator. For a locally integrable function b on \mathbb{R}^n , we define the commutator $[b, T]$ by

$$[b, T]f(x) = b(x) T f(x) - T(bf)(x)$$

for any suitable function f . Let \bar{T} be a C-Z operator. A well known result of Coifman et al. [13] states that when $K(x) = \frac{\Omega(x')}{|x|^n}$ and Ω is smooth, the commutator $[b, \bar{T}]f = b\bar{T}f - \bar{T}(bf)$ is bounded on $L_p(\mathbb{R}^n)$, $1 < p < \infty$, if and only if $b \in BMO(\mathbb{R}^n)$.

Since $BMO(\mathbb{R}^n) \subset \bigcap_{p>1} LC_p^{\{x_0\}}(\mathbb{R}^n)$, if we only assume $b \in LC_p^{\{x_0\}}(\mathbb{R}^n)$, or more generally $b \in LC_{p,\lambda}^{\{x_0\}}(\mathbb{R}^n)$, then $[b, \bar{T}]$ may not be a bounded operator on $L_p(\mathbb{R}^n)$, $1 < p < \infty$. However, it has some boundedness properties on other spaces. As a matter of fact, Grafakos et al. [24] have considered the commutator with $b \in LC_p^{\{x_0\}}(\mathbb{R}^n)$ on Herz spaces for the first time. Moreover, in [22] and [61], they have considered the commutators with $b \in LC_{p,\lambda}^{\{x_0\}}(\mathbb{R}^n)$. The commutator of C-Z operators plays an important role in studying the regularity of solutions of elliptic partial differential equations of second order (see, for example, [11, 12, 19]). The boundedness of the commutator has been generalized to other contexts and important applications to some non-linear PDEs have been given by Coifman et al. [14]. On the other hand, For $b \in L_1^{loc}(\mathbb{R}^n)$, the commutator $[b, \bar{T}_\alpha]$ of fractional integral operator (also known as the Riesz potential) is defined by

$$[b, \bar{T}_\alpha]f(x) = b(x)\bar{T}_\alpha f(x) - \bar{T}_\alpha(bf)(x) = \int_{\mathbb{R}^n} \frac{b(x) - b(y)}{|x - y|^{n-\alpha}} f(y) dy \quad 0 < \alpha < n$$

for any suitable function f .

The function b is also called the symbol function of $[b, \bar{T}_\alpha]$. The characterization of (L_p, L_q) -boundedness of the commutator $[b, \bar{T}_\alpha]$ of fractional integral operator has been given by Chanillo [8]. A well known result of Chanillo [8] states that the commutator $[b, \bar{T}_\alpha]$ is bounded from $L_p(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$, $1 < p < q < \infty$, $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n}$ if and only if $b \in BMO(\mathbb{R}^n)$. There are two major reasons for considering the problem of commutators. The first one is that the boundedness of commutators can produce some characterizations of function spaces (see [5, 8, 27, 28, 29, 30, 31, 48, 55]). The other one is that the theory of commutators plays an important role in the study of the regularity of solutions to elliptic and parabolic PDEs of the second order (see [11, 12, 19, 20, 56]).

The definition of local Campanato space $LC_{p,\lambda}^{\{x_0\}}$ as follows.

Definition 6. [5, 27, 28] Let $1 \leq p < \infty$ and $0 \leq \lambda < \frac{1}{n}$. A function $f \in L_p^{loc}(\mathbb{R}^n)$ is said to belong to the $LC_{p,\lambda}^{\{x_0\}}(\mathbb{R}^n)$ (local Campanato space), if

$$(4.1) \quad \|f\|_{LC_{p,\lambda}^{\{x_0\}}} = \sup_{r>0} \left(\frac{1}{|B(x_0, r)|^{1+\lambda p}} \int_{B(x_0, r)} |f(y) - f_{B(x_0, r)}|^p dy \right)^{\frac{1}{p}} < \infty,$$

where

$$f_{B(x_0, r)} = \frac{1}{|B(x_0, r)|} \int_{B(x_0, r)} f(y) dy.$$

Define

$$LC_{p,\lambda}^{\{x_0\}}(\mathbb{R}^n) = \left\{ f \in L_p^{loc}(\mathbb{R}^n) : \|f\|_{LC_{p,\lambda}^{\{x_0\}}} < \infty \right\}.$$

Remark 3. If two functions which differ by a constant are regarded as a function in the space $LC_{p,\lambda}^{\{x_0\}}(\mathbb{R}^n)$, then $LC_{p,\lambda}^{\{x_0\}}(\mathbb{R}^n)$ becomes a Banach space. The space $LC_{p,\lambda}^{\{x_0\}}(\mathbb{R}^n)$ when $\lambda = 0$ is just the $LC_p^{\{x_0\}}(\mathbb{R}^n)$. Apparently, (4.1) is equivalent to the following condition:

$$\sup_{r>0} \inf_{c \in \mathbb{C}} \left(\frac{1}{|B(x_0, r)|^{1+\lambda p}} \int_{B(x_0, r)} |f(y) - c|^p dy \right)^{\frac{1}{p}} < \infty.$$

In [36], Lu and Yang have introduced the central BMO space $CBMO_p(\mathbb{R}^n) = LC_{p,0}^{\{0\}}(\mathbb{R}^n)$. Also the space $CBMO^{\{x_0\}}(\mathbb{R}^n) = LC_{1,0}^{\{x_0\}}(\mathbb{R}^n)$ has been considered in other denotes in [53]. The space $LC_p^{\{x_0\}}(\mathbb{R}^n)$ can be regarded as a local version of $BMO(\mathbb{R}^n)$, the space of bounded mean oscillation, at the origin. But, they have quite different properties. The classical John-Nirenberg inequality shows that functions in $BMO(\mathbb{R}^n)$ are locally exponentially integrable. This implies that, for any $1 \leq p < \infty$, the functions in $BMO(\mathbb{R}^n)$ can be described by means of the condition:

$$\sup_{B \subset \mathbb{R}^n} \left(\frac{1}{|B|} \int_B |f(y) - f_B|^p dy \right)^{\frac{1}{p}} < \infty,$$

where B denotes an arbitrary ball in \mathbb{R}^n . However, the space $LC_p^{\{x_0\}}(\mathbb{R}^n)$ depends on p . If $p_1 < p_2$, then $LC_{p_2}^{\{x_0\}}(\mathbb{R}^n) \subsetneq LC_{p_1}^{\{x_0\}}(\mathbb{R}^n)$. Therefore, there is no analogy of the famous John-Nirenberg inequality of $BMO(\mathbb{R}^n)$ for the space $LC_p^{\{x_0\}}(\mathbb{R}^n)$. One can imagine that the behavior of $LC_p^{\{x_0\}}(\mathbb{R}^n)$ may be quite different from that of $BMO(\mathbb{R}^n)$ (see [39]).

Theorem 6. [5, 27, 28, 29, 30] Let v_1, v_2 and ω be weights on $(0, \infty)$ and $v_1(t)$ be bounded outside a neighbourhood of the origin. The inequality

$$(4.2) \quad \operatorname{esssup}_{r>0} v_2(r) H_{\omega}^* g(r) \leq C \operatorname{esssup}_{r>0} v_1(r) g(r)$$

holds for some $C > 0$ for all non-negative and non-decreasing functions g on $(0, \infty)$ if and only if

$$(4.3) \quad B := \sup_{r>0} v_2(r) \int_r^\infty \left(1 + \ln \frac{t}{r}\right) \frac{\omega(t)dt}{\operatorname{esssup}_{t<s<\infty} v_1(s)} < \infty.$$

Moreover, the value $C = B$ is the best constant for (4.2).

Remark 4. In (4.2) and (4.3) it is assumed that $\frac{1}{\infty} = 0$ and $0 \cdot \infty = 0$.

Lemma 3. Let b be function in $LC_{p,\lambda}^{\{x_0\}}(\mathbb{R}^n)$, $1 \leq p < \infty$, $0 \leq \lambda < \frac{1}{n}$ and $r_1, r_2 > 0$. Then

$$(4.4) \quad \left(\frac{1}{|B(x_0, r_1)|^{1+\lambda p}} \int_{B(x_0, r_1)} |b(y) - b_{B(x_0, r_2)}|^p dy \right)^{\frac{1}{p}} \leq C \left(1 + \ln \frac{r_1}{r_2}\right) \|b\|_{LC_{p,\lambda}^{\{x_0\}}},$$

where $C > 0$ is independent of b, r_1 and r_2 .

From this inequality (4.4), we have

$$(4.5) \quad |b_{B(x_0, r_1)} - b_{B(x_0, r_2)}| \leq C \left(1 + \ln \frac{r_1}{r_2}\right) |B(x_0, r_1)|^\lambda \|b\|_{LC_{p,\lambda}^{\{x_0\}}},$$

and it is easy to see that

$$(4.6) \quad \|b - (b)_B\|_{L_p(B)} \leq C \left(1 + \ln \frac{r_1}{r_2}\right) r^{\frac{n}{p} + n\lambda} \|b\|_{LC_{p,\lambda}^{\{x_0\}}}.$$

As in the proof of Theorem 4, it suffices to prove the following Lemma 4.

Lemma 4. Let $x_0 \in \mathbb{R}^n$, $\Omega \in L_s(S^{n-1})$, $1 < s \leq \infty$, be homogeneous of degree zero. Let $T_{\Omega, \alpha}$ be a linear operator satisfying condition (1.1), bounded from $L_p(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$. Let also $0 < \alpha < n$, $1 < p < \frac{n}{\alpha}$, $b \in LC_{p_2, \lambda}^{\{x_0\}}(\mathbb{R}^n)$, $0 \leq \lambda < \frac{1}{n}$, $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$, $\frac{1}{q_1} = \frac{1}{p_1} - \frac{\alpha}{n}$. Then, for $s' \leq p$ the inequality

$$(4.7) \quad \|T_{\Omega, b, \alpha} f\|_{L_q(B(x_0, r))} \lesssim \|b\|_{LC_{p_2, \lambda}^{\{x_0\}}} r^{\frac{n}{q}} \int_{2r}^\infty \left(1 + \ln \frac{t}{r}\right) t^{n\lambda - \frac{n}{q_1} - 1} \|f\|_{L_{p_1}(B(x_0, t))} dt$$

holds for any ball $B(x_0, r)$ and for all $f \in L_{p_1}^{loc}(\mathbb{R}^n)$.

Also, for $q_1 < s$ the inequality

$$\|T_{\Omega, b, \alpha} f\|_{L_q(B(x_0, r))} \lesssim \|b\|_{LC_{p_2, \lambda}^{\{x_0\}}} r^{\frac{n}{q} - \frac{n}{s}} \int_{2r}^\infty \left(1 + \ln \frac{t}{r}\right) t^{n\lambda - \frac{n}{q_1} + \frac{n}{s} - 1} \|f\|_{L_{p_1}(B(x_0, t))} dt$$

holds for any ball $B(x_0, r)$ and for all $f \in L_{p_1}^{loc}(\mathbb{R}^n)$.

Proof. Let $1 < p < \infty$, $0 < \alpha < \frac{n}{p}$, $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$ and $\frac{1}{q_1} = \frac{1}{p_1} - \frac{\alpha}{n}$. As in the proof of Lemma 2, we represent f in form (3.5) and have

$$\begin{aligned} T_{\Omega, b, \alpha} f(x) &= (b(x) - b_B) T_{\Omega, \alpha} f_1(x) - T_{\Omega, \alpha} ((b(\cdot) - b_B) f_1)(x) \\ &\quad + (b(x) - b_B) T_{\Omega, \alpha} f_2(x) - T_{\Omega, \alpha} ((b(\cdot) - b_B) f_2)(x) \\ &\equiv J_1 + J_2 + J_3 + J_4. \end{aligned}$$

Hence we get

$$\|T_{\Omega,b,\alpha}f\|_{L_q(B)} \leq \|J_1\|_{L_q(B)} + \|J_2\|_{L_q(B)} + \|J_3\|_{L_q(B)} + \|J_4\|_{L_q(B)}.$$

By the Hölder's inequality, the boundedness of $T_{\Omega,\alpha}$ from $L_{p_1}(\mathbb{R}^n)$ to $L_{q_1}(\mathbb{R}^n)$ (see Lemma 1) it follows that:

$$\begin{aligned} \|J_1\|_{L_q(B)} &\leq \|(b(\cdot) - b_B)T_{\Omega,\alpha}f_1(\cdot)\|_{L_q(\mathbb{R}^n)} \\ &\lesssim \|(b(\cdot) - b_B)\|_{L_{p_2}(\mathbb{R}^n)} \|T_{\Omega,\alpha}f_1(\cdot)\|_{L_{q_1}(\mathbb{R}^n)} \\ &\lesssim \|b\|_{LC_{p_2,\lambda}^{\{x_0\}}} r^{\frac{n}{p_2}+n\lambda} \|f_1\|_{L_{p_1}(\mathbb{R}^n)} \\ &= \|b\|_{LC_{p_2,\lambda}^{\{x_0\}}} r^{\frac{n}{p_2}+\frac{n}{q_1}+n\lambda} \|f\|_{L_{p_1}(2B)} \int_{2r}^{\infty} t^{-1-\frac{n}{q_1}} dt \\ &\lesssim \|b\|_{LC_{p_2,\lambda}^{\{x_0\}}} r^{\frac{n}{q}+n\lambda} \int_{2r}^{\infty} \left(1 + \ln \frac{t}{r}\right) \|f\|_{L_{p_1}(B(x_0,t))} t^{-1-\frac{n}{q_1}} dt. \end{aligned}$$

Using the the boundedness of $T_{\Omega,\alpha}$ from $L_p(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$ (see Lemma 1), by the Hölder's inequality for J_2 , we have

$$\begin{aligned} \|J_2\|_{L_q(B)} &\leq \|T_{\Omega,\alpha}^P(b(\cdot) - b_B)f_1\|_{L_q(\mathbb{R}^n)} \\ &\lesssim \|(b(\cdot) - b_B)f_1\|_{L_p(\mathbb{R}^n)} \\ &\lesssim \|b(\cdot) - b_B\|_{L_{p_2}(\mathbb{R}^n)} \|f_1\|_{L_{p_1}(\mathbb{R}^n)} \\ &\lesssim \|b\|_{LC_{p_2,\lambda}^{\{x_0\}}} r^{\frac{n}{p_2}+\frac{n}{q_1}+n\lambda} \|f\|_{L_{p_1}(2B)} \int_{2r}^{\infty} t^{-1-\frac{n}{q_1}} dt \\ &\lesssim \|b\|_{LC_{p_2,\lambda}^{\{x_0\}}} r^{\frac{n}{q}+n\lambda} \int_{2r}^{\infty} \left(1 + \ln \frac{t}{r}\right) \|f\|_{L_{p_1}(B(x_0,t))} t^{-1-\frac{n}{q_1}} dt. \end{aligned}$$

For J_3 , it is known that $x \in B$, $y \in (2B)^C$, which implies $\frac{1}{2}|x_0 - y| \leq |x - y| \leq \frac{3}{2}|x_0 - y|$.

When $s' \leq p_1$, by the Fubini's theorem, the Hölder's inequality and (3.7), we have

$$\begin{aligned}
|T_{\Omega, \alpha} f_2(x)| &\leq c_0 \int_{(2B)^C} |\Omega(x-y)| \frac{|f(y)|}{|x_0-y|^{n-\alpha}} dy \\
&\approx \int_{2r}^{\infty} \int_{2r < |x_0-y| < t} |\Omega(x-y)| |f(y)| dy t^{-1-n+\alpha} dt \\
&\lesssim \int_{2r}^{\infty} \int_{B(x_0, t)} |\Omega(x-y)| |f(y)| dy t^{-1-n+\alpha} dt \\
&\lesssim \int_{2r}^{\infty} \|f\|_{L_{p_1}(B(x_0, t))} \|\Omega(x-\cdot)\|_{L_s(B(x_0, t))} |B(x_0, t)|^{1-\frac{1}{p_1}-\frac{1}{s}} t^{-1-n+\alpha} dt \\
&\lesssim \int_{2r}^{\infty} \|f\|_{L_{p_1}(B(x_0, t))} t^{-1-\frac{n}{q_1}} dt.
\end{aligned}$$

Hence, we get

$$\begin{aligned}
\|J_3\|_{L_q(B)} &\leq \|(b(\cdot) - b_B) T_{\Omega, \alpha} f_2(\cdot)\|_{L_q(\mathbb{R}^n)} \\
&\lesssim \|(b(\cdot) - b_B)\|_{L_q(\mathbb{R}^n)} \int_{2r}^{\infty} \|f\|_{L_{p_1}(B(x_0, t))} t^{-1-\frac{n}{q_1}} dt \\
&\lesssim \|(b(\cdot) - b_B)\|_{L_{p_2}(\mathbb{R}^n)} r^{\frac{n}{q_1}} \int_{2r}^{\infty} \|f\|_{L_{p_1}(B(x_0, t))} t^{-1-\frac{n}{q_1}} dt \\
&\lesssim \|b\|_{LC_{p_2, \lambda}^{\{x_0\}}} r^{\frac{n}{q} + n\lambda} \int_{2r}^{\infty} \left(1 + \ln \frac{t}{r}\right) \|f\|_{L_{p_1}(B(x_0, t))} t^{-1-\frac{n}{q_1}} dt \\
&\lesssim \|b\|_{LC_{p_2, \lambda}^{\{x_0\}}} r^{\frac{n}{q}} \int_{2r}^{\infty} \left(1 + \ln \frac{t}{r}\right) t^{n\lambda - \frac{n}{q_1} - 1} \|f\|_{L_{p_1}(B(x_0, t))} dt.
\end{aligned}$$

When $q_1 < s$, by the Fubini's theorem, the Minkowski inequality, the Hölder's inequality and from (4.6), (3.10), we get

$$\begin{aligned}
\|J_3\|_{L_q(B)} &\leq \left(\int_B \left| \int_{2r}^{\infty} \int_{B(x_0,t)} |f(y)| |b(x) - b_B| |\Omega(x-y)| dy \frac{dt}{t^{n-\alpha+1}} \right|^q dx \right)^{\frac{1}{q}} \\
&\leq \int_{2r}^{\infty} \int_{B(x_0,t)} |f(y)| \| (b(\cdot) - b_B) \Omega(\cdot - y) \|_{L_q(B)} dy \frac{dt}{t^{n-\alpha+1}} \\
&\leq \int_{2r}^{\infty} \int_{B(x_0,t)} |f(y)| \|b(\cdot) - b_B\|_{L_{p_2}(B)} \|\Omega(\cdot - y)\|_{L_{q_1}(B)} dy \frac{dt}{t^{n-\alpha+1}} \\
&\lesssim \|b\|_{LC_{p_2,\lambda}^{\{x_0\}}} r^{\frac{n}{p_2} + n\lambda} |B|^{\frac{1}{q_1} - \frac{1}{s}} \int_{2r}^{\infty} \int_{B(x_0,t)} |f(y)| \|\Omega(\cdot - y)\|_{L_s(B)} dy \frac{dt}{t^{n-\alpha+1}} \\
&\lesssim \|b\|_{LC_{p_2,\lambda}^{\{x_0\}}} r^{\frac{n}{q} - \frac{n}{s} + n\lambda} \int_{2r}^{\infty} \|f\|_{L_1(B(x_0,t))} \left| B\left(x_0, \frac{3}{2}t\right) \right|^{\frac{1}{s}} \frac{dt}{t^{n-\alpha+1}} \\
&\lesssim \|b\|_{LC_{p_2,\lambda}^{\{x_0\}}} r^{\frac{n}{q} - \frac{n}{s} + n\lambda} \int_{2r}^{\infty} \left(1 + \ln \frac{t}{r}\right) \|f\|_{L_{p_1}(B(x_0,t))} \frac{dt}{t^{\frac{n}{q_1} - \frac{n}{s} + 1}} \\
&\lesssim \|b\|_{LC_{p_2,\lambda}^{\{x_0\}}} r^{\frac{n}{q} - \frac{n}{s}} \int_{2r}^{\infty} \left(1 + \ln \frac{t}{r}\right) t^{n\lambda - \frac{n}{q_1} + \frac{n}{s} - 1} \|f\|_{L_{p_1}(B(x_0,t))} dt.
\end{aligned}$$

On the other hand, for J_4 , when $s' \leq p$, for $x \in B$, by the Fubini's theorem, applying the Hölder's inequality and from (4.5), (4.6) (3.7) we have

$$\begin{aligned}
|T_{\Omega,\alpha}((b(\cdot) - b_B) f_2)(x)| &\lesssim \int_{(2B)^C} |b(y) - b_B| |\Omega(x-y)| \frac{|f(y)|}{|x-y|^{n-\alpha}} dy \\
&\lesssim \int_{(2B)^C} |b(y) - b_B| |\Omega(x-y)| \frac{|f(y)|}{|x_0-y|^{n-\alpha}} dy \\
&\approx \int_{2r}^{\infty} \int_{2r < |x_0-y| < t} |b(y) - b_B| |\Omega(x-y)| |f(y)| dy \frac{dt}{t^{n-\alpha+1}} \\
&\lesssim \int_{2r}^{\infty} \int_{B(x_0,t)} |b(y) - b_{B(x_0,t)}| |\Omega(x-y)| |f(y)| dy \frac{dt}{t^{n-\alpha+1}} \\
&\quad + \int_{2r}^{\infty} |b_{B(x_0,r)} - b_{B(x_0,t)}| \int_{B(x_0,t)} |\Omega(x-y)| |f(y)| dy \frac{dt}{t^{n-\alpha+1}} \\
&\lesssim \int_{2r}^{\infty} \| (b(\cdot) - b_{B(x_0,t)}) f \|_{L_p(B(x_0,t))} \|\Omega(\cdot - y)\|_{L_s(B(x_0,t))} |B(x_0,t)|^{1-\frac{1}{p}-\frac{1}{s}} \frac{dt}{t^{n-\alpha+1}}
\end{aligned}$$

$$\begin{aligned}
& + \int_{2r}^{\infty} |b_{B(x_0, r)} - b_{B(x_0, t)}| \|f\|_{L_{p_1}(B(x_0, t))} \|\Omega(\cdot - y)\|_{L_s(B(x_0, t))} |B(x_0, t)|^{1 - \frac{1}{p_1} - \frac{1}{s}} t^{\alpha - n - 1} dt \\
& \lesssim \int_{2r}^{\infty} \|(b(\cdot) - b_{B(x_0, t)})\|_{L_{p_2}(B(x_0, t))} \|f\|_{L_{p_1}(B(x_0, t))} t^{-1 - \frac{n}{q}} dt \\
& + \|b\|_{LC_{p_2, \lambda}^{\{x_0\}}} \int_{2r}^{\infty} \left(1 + \ln \frac{t}{r}\right) \|f\|_{L_{p_1}(B(x_0, t))} t^{-1 - \frac{n}{q_1} + n\lambda} dt \\
& \lesssim \|b\|_{LC_{p_2, \lambda}^{\{x_0\}}} \int_{2r}^{\infty} \left(1 + \ln \frac{t}{r}\right) \|f\|_{L_{p_1}(B(x_0, t))} t^{-1 - \frac{n}{q_1} + n\lambda} dt.
\end{aligned}$$

Then, we have

$$\begin{aligned}
\|J_4\|_{L_q(B)} & = \|T_{\Omega, \alpha}(b(\cdot) - b_B) f_2\|_{L_q(B)} \\
& \lesssim \|b\|_{LC_{p_2, \lambda}^{\{x_0\}}} r^{\frac{n}{q}} \int_{2r}^{\infty} \left(1 + \ln \frac{t}{r}\right) t^{n\lambda - \frac{n}{q_1} - 1} \|f\|_{L_{p_1}(B(x_0, t))} dt.
\end{aligned}$$

When $q < s$, by the Fubini's theorem, the Minkowski inequality, the Hölder's inequality and from (4.5), (4.6), (3.10) we have

$$\begin{aligned}
\|J_4\|_{L_q(B)} &\leq \left(\int_B \left| \int_{2r}^{\infty} \int_{B(x_0,t)} |b(y) - b_{B(x_0,t)}| |f(y)| |\Omega(x-y)| dy \frac{dt}{t^{n-\alpha+1}} \right|^q dx \right)^{\frac{1}{q}} \\
&\quad + \left(\int_B \left| \int_{2r}^{\infty} |b_{B(x_0,r)} - b_{B(x_0,t)}| \int_{B(x_0,t)} |f(y)| |\Omega(x-y)| dy \frac{dt}{t^{n-\alpha+1}} \right|^q dx \right)^{\frac{1}{q}} \\
&\lesssim \int_{2r}^{\infty} \int_{B(x_0,t)} |b(y) - b_{B(x_0,t)}| |f(y)| \|\Omega(\cdot - y)\|_{L_q(B(x_0,t))} dy \frac{dt}{t^{n-\alpha+1}} \\
&\quad + \int_{2r}^{\infty} |b_{B(x_0,r)} - b_{B(x_0,t)}| \int_{B(x_0,t)} |f(y)| \|\Omega(\cdot - y)\|_{L_q(B(x_0,t))} dy \frac{dt}{t^{n-\alpha+1}} \\
&\lesssim |B|^{\frac{1}{q} - \frac{1}{s}} \int_{2r}^{\infty} \int_{B(x_0,t)} |b(y) - b_{B(x_0,t)}| |f(y)| \|\Omega(\cdot - y)\|_{L_s(B(x_0,t))} dy \frac{dt}{t^{n-\alpha+1}} \\
&\quad + |B|^{\frac{1}{q} - \frac{1}{s}} \int_{2r}^{\infty} |b_{B(x_0,r)} - b_{B(x_0,t)}| \int_{B(x_0,t)} |f(y)| \|\Omega(\cdot - y)\|_{L_s(B(x_0,t))} dy \frac{dt}{t^{n-\alpha+1}} \\
&\lesssim r^{\frac{n}{q} - \frac{n}{s}} \int_{2r}^{\infty} \|b(\cdot) - b_{B(x_0,t)}\|_{L_{p_2}(B(x_0,t))} \|f\|_{L_{p_1}(B(x_0,t))} |B(x_0,t)|^{1-\frac{1}{p}} \left| B\left(x_0, \frac{3}{2}t\right) \right|^{\frac{1}{s}} \frac{dt}{t^{n-\alpha+1}} \\
&\quad + r^{\frac{n}{q} - \frac{n}{s}} \int_{2r}^{\infty} |b_{B(x_0,r)} - b_{B(x_0,t)}| \|f\|_{L_{p_1}(B(x_0,t))} \left| B\left(x_0, \frac{3}{2}t\right) \right|^{\frac{1}{s}} \frac{dt}{t^{\frac{n}{p_1}-\alpha+1}} \\
&\lesssim r^{\frac{n}{q} - \frac{n}{s}} \|b\|_{LC_{p_2,\lambda}^{\{x_0\}}} \int_{2r}^{\infty} \left(1 + \ln \frac{t}{r}\right) t^{n\lambda - \frac{n}{q_1} + \frac{n}{s} - 1} \|f\|_{L_{p_1}(B(x_0,t))} dt.
\end{aligned}$$

By combining the above estimates, we complete the proof of Lemma 4. \square

Now we can give the following theorem (our main result).

Theorem 7. Let $x_0 \in \mathbb{R}^n$, $\Omega \in L_s(S^{n-1})$, $1 < s \leq \infty$, be homogeneous of degree zero. Let $T_{\Omega,\alpha}$ be a linear operator satisfying condition (1.1) and bounded from $L_p(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$. Let $0 < \alpha < n$, $1 < p < \frac{n}{\alpha}$, $b \in LC_{p_2,\lambda}^{\{x_0\}}(\mathbb{R}^n)$, $0 \leq \lambda < \frac{1}{n}$, $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$, $\frac{1}{q_1} = \frac{1}{p_1} - \frac{\alpha}{n}$.

Let also, for $s' \leq p$ the pair (φ_1, φ_2) satisfies the condition

$$(4.8) \quad \int_r^\infty \left(1 + \ln \frac{t}{r}\right) \frac{\operatorname{ess\,inf}_{t < \tau < \infty} \varphi_1(x_0, \tau) \tau^{\frac{n}{p_1}}}{t^{\frac{n}{q_1} + 1 - n\lambda}} dt \leq C \varphi_2(x_0, r),$$

and for $q_1 < s$ the pair (φ_1, φ_2) satisfies the condition

$$(4.9) \quad \int_r^\infty \left(1 + \ln \frac{t}{r}\right) \frac{\operatorname{ess\,inf}_{t < \tau < \infty} \varphi_1(x_0, \tau) \tau^{\frac{n}{p_1}}}{t^{\frac{n}{q_1} - \frac{n}{s} + 1 - n\lambda}} dt \leq C \varphi_2(x_0, r) r^{\frac{n}{s}},$$

where C does not depend on r .

Then, the operator $T_{\Omega, b, \alpha}$ is bounded from $LM_{p_1, \varphi_1}^{\{x_0\}}$ to $LM_{q, \varphi_2}^{\{x_0\}}$. Moreover,

$$\|T_{\Omega, b, \alpha}\|_{LM_{q, \varphi_2}^{\{x_0\}}} \lesssim \|b\|_{LC_{p_2, \lambda}^{\{x_0\}}} \|f\|_{LM_{p_1, \varphi_1}^{\{x_0\}}}.$$

Proof. The statement of Theorem 7 follows by Lemma 4 and Theorem 6 in the same manner as in the proof of Theorem 4. \square

For the sublinear commutator of the fractional maximal operator with rough kernel which is defined as follows

$$M_{\Omega, b, \alpha}(f)(x) = \sup_{t > 0} |B(x, t)|^{-1 + \frac{\alpha}{n}} \int_{B(x, t)} |b(x) - b(y)| |\Omega(x - y)| |f(y)| dy$$

by Theorem 7 we get the following new result.

Corollary 6. Suppose that $x_0 \in \mathbb{R}^n$, $\Omega \in L_s(S^{n-1})$, $1 < s \leq \infty$, is homogeneous of degree zero. Let $0 < \alpha < n$, $1 < p < \frac{n}{\alpha}$, $b \in LC_{p_2, \lambda}^{\{x_0\}}(\mathbb{R}^n)$, $0 \leq \lambda < \frac{1}{n}$, $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$, $\frac{1}{q_1} = \frac{1}{p_1} - \frac{\alpha}{n}$. Let also, for $s' \leq p$ the pair (φ_1, φ_2) satisfies the condition (4.8) and for $q_1 < s$ the pair (φ_1, φ_2) satisfies the condition (4.9). Then, the operators $M_{\Omega, b, \alpha}$ and $[b, \overline{T}_{\Omega, \alpha}]$ are bounded from $LM_{p_1, \varphi_1}^{\{x_0\}}$ to $LM_{q, \varphi_2}^{\{x_0\}}$.

For the sublinear commutator of the fractional maximal operator is defined as follows

$$M_{b, \alpha}(f)(x) = \sup_{t > 0} |B(x, t)|^{-1 + \frac{\alpha}{n}} \int_{B(x, t)} |b(x) - b(y)| |f(y)| dy$$

by Theorem 7 we get the following new result.

Corollary 7. Let $x_0 \in \mathbb{R}^n$, $0 < \alpha < n$, $1 < p < \frac{n}{\alpha}$, $b \in LC_{p_2, \lambda}^{\{x_0\}}(\mathbb{R}^n)$, $0 \leq \lambda < \frac{1}{n}$, $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$, $\frac{1}{q_1} = \frac{1}{p_1} - \frac{\alpha}{n}$ and the pair (φ_1, φ_2) satisfies condition (4.8). Then, the operators $M_{b, \alpha}$ and $[b, \overline{T}_\alpha]$ are bounded from $LM_{p_1, \varphi_1}^{\{x_0\}}$ to $LM_{q, \varphi_2}^{\{x_0\}}$.

Now using above results, we also obtain the boundedness of the operator $T_{\Omega, b, \alpha}$ on the generalized vanishing local Morrey spaces $VLM_{p, \varphi}^{\{x_0\}}$.

Theorem 8. (Our main result) Let $x_0 \in \mathbb{R}^n$, $\Omega \in L_s(S^{n-1})$, $1 < s \leq \infty$, be homogeneous of degree zero. Let $0 < \alpha < n$, $1 < p < \frac{n}{\alpha}$, $b \in LC_{p_2, \lambda}^{\{x_0\}}(\mathbb{R}^n)$, $0 \leq \lambda < \frac{1}{n}$, $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$, $\frac{1}{q_1} = \frac{1}{p_1} - \frac{\alpha}{n}$, and $T_{\Omega, \alpha}$ is a linear operator satisfying condition (1.1) and bounded from $L_p(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$. Let for $s' \leq p$ the pair (φ_1, φ_2) satisfies conditions (2.5)-(2.6) and

$$(4.10) \quad \int_r^\infty \left(1 + \ln \frac{t}{r}\right) \varphi_1(x_0, t) \frac{t^{\frac{n}{p_1}}}{t^{\frac{n}{q_1} + 1 - n\lambda}} dt \leq C_0 \varphi_2(x_0, r),$$

where C_0 does not depend on $r > 0$,

$$(4.11) \quad \lim_{r \rightarrow 0} \frac{\ln \frac{1}{r}}{\varphi_2(x_0, r)} = 0$$

and

$$(4.12) \quad c_\delta := \int_{\delta}^{\infty} (1 + \ln |t|) \varphi_1(x_0, t) \frac{t^{\frac{n}{p_1}}}{t^{\frac{n}{q_1} + 1 - n\lambda}} dt < \infty$$

for every $\delta > 0$, and for $q_1 < s$ the pair (φ_1, φ_2) satisfies conditions (2.5)-(2.6) and also

$$(4.13) \quad \int_r^{\infty} \left(1 + \ln \frac{t}{r}\right) \varphi_1(x_0, t) \frac{t^{\frac{n}{p_1}}}{t^{\frac{n}{q_1} - \frac{n}{s} + 1 - n\lambda}} dt \leq C_0 \varphi_2(x_0, r) r^{\frac{n}{s}},$$

where C_0 does not depend on $r > 0$,

$$\lim_{r \rightarrow 0} \frac{\ln \frac{1}{r}}{\varphi_2(x_0, r)} = 0$$

and

$$(4.14) \quad c_{\delta'} := \int_{\delta'}^{\infty} (1 + \ln |t|) \varphi_1(x_0, t) \frac{t^{\frac{n}{p_1}}}{t^{\frac{n}{q_1} - \frac{n}{s} + 1 - n\lambda}} dt < \infty$$

for every $\delta' > 0$.

Then the operator $T_{\Omega, b, \alpha}$ is bounded from $VL M_{p_1, \varphi_1}^{\{x_0\}}$ to $VL M_{q, \varphi_2}^{\{x_0\}}$. Moreover,

$$(4.15) \quad \|T_{\Omega, b, \alpha} f\|_{VL M_{q, \varphi_2}^{\{x_0\}}} \lesssim \|b\|_{LC_{p_2, \lambda}^{\{x_0\}}} \|f\|_{VL M_{p_1, \varphi_1}^{\{x_0\}}}.$$

Proof. The norm inequality having already been provided by Theorem 7, we only have to prove the implication

$$(4.16) \quad \lim_{r \rightarrow 0} \frac{r^{-\frac{n}{p_1}} \|f\|_{L_{p_1}(B(x_0, r))}}{\varphi_1(x_0, r)} = 0 \text{ implies } \lim_{r \rightarrow 0} \frac{r^{-\frac{n}{q}} \|T_{\Omega, b, \alpha} f\|_{L_q(B(x_0, r))}}{\varphi_2(x_0, r)} = 0.$$

To show that

$$\frac{r^{-\frac{n}{q}} \|T_{\Omega, b, \alpha} f\|_{L_q(B(x_0, r))}}{\varphi_2(x_0, r)} < \epsilon \text{ for small } r,$$

we use the estimate (4.7):

$$\frac{r^{-\frac{n}{q}} \|T_{\Omega, b, \alpha} f\|_{L_q(B(x_0, r))}}{\varphi_2(x_0, r)} \lesssim \frac{\|b\|_{LC_{p_2, \lambda}^{\{x_0\}}}}{\varphi_2(x_0, r)} \int_r^{\infty} \left(1 + \ln \frac{t}{r}\right) t^{n\lambda - \frac{n}{q_1} - 1} \|f\|_{L_{p_1}(B(x_0, t))} dt.$$

We take $r < \delta_0$, where δ_0 will be chosen small enough and split the integration:

$$(4.17) \quad \frac{r^{-\frac{n}{q}} \|T_{\Omega, b, \alpha} f\|_{L_q(B(x_0, r))}}{\varphi_2(x_0, r)} \leq C [I_{\delta_0}(x_0, r) + J_{\delta_0}(x_0, r)],$$

where $\delta_0 > 0$ (we may take $\delta_0 < 1$), and

$$I_{\delta_0}(x_0, r) := \frac{1}{\varphi_2(x_0, r)} \int_r^{\delta_0} \left(1 + \ln \frac{t}{r}\right) t^{n\lambda - \frac{n}{q_1} - 1} \|f\|_{L_{p_1}(B(x_0, t))} dt,$$

and

$$J_{\delta_0}(x_0, r) := \frac{1}{\varphi_2(x_0, r)} \int_{\delta_0}^{\infty} \left(1 + \ln \frac{t}{r}\right) t^{n\lambda - \frac{n}{q_1} - 1} \|f\|_{L_{p_1}(B(x_0, t))} dt$$

and $r < \delta_0$. Now we choose any fixed $\delta_0 > 0$ such that

$$\frac{t^{-\frac{n}{p_1}} \|f\|_{L_{p_1}(B(x_0, t))}}{\varphi_1(x_0, t)} < \frac{\epsilon}{2CC_0}, \quad t \leq \delta_0,$$

where C and C_0 are constants from (4.10) and (4.17). This allows to estimate the first term uniformly in $r \in (0, \delta_0)$:

$$CI_{\delta_0}(x_0, r) < \frac{\epsilon}{2}, \quad 0 < r < \delta_0.$$

For the second term, writing $1 + \ln \frac{t}{r} \leq 1 + |\ln t| + \ln \frac{1}{r}$, we obtain

$$J_{\delta_0}(x_0, r) \leq \frac{c_{\delta_0} + \widetilde{c_{\delta_0}} \ln \frac{1}{r}}{\varphi_2(x_0, r)} \|f\|_{LM_{p_1, \varphi_1}^{\{x_0\}}},$$

where c_{δ_0} is the constant from (4.12) with $\delta = \delta_0$ and $\widetilde{c_{\delta_0}}$ is a similar constant with omitted logarithmic factor in the integrand. Then, by (4.11) we can choose small enough r such that

$$J_{\delta_0}(x_0, r) < \frac{\epsilon}{2},$$

which completes the proof of (4.16).

For the case of $q_1 < s$, we can also use the same method, so we omit the details. \square

Remark 5. Conditions (4.12) and (4.14) are not needed in the case when $\varphi(x_0, r)$ does not depend on x_0 , since (4.12) follows from (4.10) and similarly, (4.14) follows from (4.13) in this case.

Corollary 8. Suppose that $x_0 \in \mathbb{R}^n$, $\Omega \in L_s(S^{n-1})$, $1 < s \leq \infty$, is homogeneous of degree zero. Let $0 < \alpha < n$, $1 < p < \frac{n}{\alpha}$, $b \in LC_{p_2, \lambda}^{\{x_0\}}(\mathbb{R}^n)$, $0 \leq \lambda < \frac{1}{n}$, $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$, $\frac{1}{q_1} = \frac{1}{p_1} - \frac{\alpha}{n}$. If for $s' \leq p$ the pair (φ_1, φ_2) satisfies conditions (2.5)-(2.6)-(4.11) and (4.12)-(4.10) and for $q_1 < s$ the pair (φ_1, φ_2) satisfies conditions (2.5)-(2.6)-(4.11) and (4.14)-(4.13). Then, the operators $M_{\Omega, b, \alpha}$ and $[b, \overline{T}_{\Omega, \alpha}]$ are bounded from $VLM_{p_1, \varphi_1}^{\{x_0\}}$ to $VLM_{q, \varphi_2}^{\{x_0\}}$.

In the case of $q = \infty$ by Theorem 8, we get

Corollary 9. Let $x_0 \in \mathbb{R}^n$, $0 < \alpha < n$, $1 < p < \frac{n}{\alpha}$, $b \in LC_{p_2, \lambda}^{\{x_0\}}(\mathbb{R}^n)$, $0 \leq \lambda < \frac{1}{n}$, $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$, $\frac{1}{q_1} = \frac{1}{p_1} - \frac{\alpha}{n}$ and the pair (φ_1, φ_2) satisfies conditions (2.5)-(2.6)-(4.11) and (4.12)-(4.10). Then the operators $M_{b, \alpha}$ and $[b, \overline{T}_{\alpha}]$ are bounded from $VLM_{p_1, \varphi_1}^{\{x_0\}}$ to $VLM_{q, \varphi_2}^{\{x_0\}}$.

5. SOME APPLICATIONS

In this section, we give the applications of Theorem 4, Theorem 5, Theorem 7, Theorem 8 for the Marcinkiewicz operator.

5.1. Marcinkiewicz Operator. Let $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$ be the unit sphere in \mathbb{R}^n equipped with the Lebesgue measure $d\sigma$. Suppose that Ω satisfies the following conditions.

(a) Ω is the homogeneous function of degree zero on $\mathbb{R}^n \setminus \{0\}$, that is,

$$\Omega(\mu x) = \Omega(x), \text{ for any } \mu > 0, x \in \mathbb{R}^n \setminus \{0\}.$$

(b) Ω has mean zero on S^{n-1} , that is,

$$\int_{S^{n-1}} \Omega(x') d\sigma(x') = 0,$$

where $x' = \frac{x}{|x|}$ for any $x \neq 0$.

(c) $\Omega \in Lip_\gamma(S^{n-1})$, $0 < \gamma \leq 1$, that is there exists a constant $M > 0$ such that,

$$|\Omega(x') - \Omega(y')| \leq M|x' - y'|^\gamma \text{ for any } x', y' \in S^{n-1}.$$

In 1958, Stein [58] defined the Marcinkiewicz integral of higher dimension μ_Ω as

$$\mu_\Omega(f)(x) = \left(\int_0^\infty |F_{\Omega,t}(f)(x)|^2 \frac{dt}{t^3} \right)^{1/2},$$

where

$$F_{\Omega,t}(f)(x) = \int_{|x-y| \leq t} \frac{\Omega(x-y)}{|x-y|^{n-1}} f(y) dy.$$

Since Stein's work in 1958, the continuity of Marcinkiewicz integral has been extensively studied as a research topic and also provides useful tools in harmonic analysis [38, 59, 60, 62].

The Marcinkiewicz operator is defined by (see [63])

$$\mu_{\Omega,\alpha}(f)(x) = \left(\int_0^\infty |F_{\Omega,\alpha,t}(f)(x)|^2 \frac{dt}{t^3} \right)^{1/2},$$

where

$$F_{\Omega,\alpha,t}(f)(x) = \int_{|x-y| \leq t} \frac{\Omega(x-y)}{|x-y|^{n-1-\alpha}} f(y) dy.$$

Note that $\mu_\Omega f = \mu_{\Omega,0} f$.

The sublinear commutator of the operator $\mu_{\Omega,\alpha}$ is defined by

$$[b, \mu_{\Omega,\alpha}](f)(x) = \left(\int_0^\infty |F_{\Omega,\alpha,t,b}(f)(x)|^2 \frac{dt}{t^3} \right)^{1/2},$$

where

$$F_{\Omega,\alpha,t,b}(f)(x) = \int_{|x-y| \leq t} \frac{\Omega(x-y)}{|x-y|^{n-1-\alpha}} [b(x) - b(y)] f(y) dy.$$

We consider the space $H = \{h : \|h\| = (\int_0^\infty |h(t)|^2 \frac{dt}{t^3})^{1/2} < \infty\}$. Then, it is clear that $\mu_{\Omega,\alpha}(f)(x) = \|F_{\Omega,\alpha,t}(x)\|$.

By the Minkowski inequality, we get

$$\mu_{\Omega,\alpha}(f)(x) \leq \int_{\mathbb{R}^n} \frac{|\Omega(x-y)|}{|x-y|^{n-1-\alpha}} |f(y)| \left(\int_{|x-y|}^{\infty} \frac{dt}{t^3} \right)^{1/2} dy \leq C \int_{\mathbb{R}^n} \frac{|\Omega(x-y)|}{|x-y|^{n-\alpha}} |f(y)| dy.$$

Thus, $\mu_{\Omega,\alpha}$ satisfies the condition (1.1). It is known that $\mu_{\Omega,\alpha}$ is bounded from $L_p(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$ for $p > 1$, and bounded from $L_1(\mathbb{R}^n)$ to $WL_q(\mathbb{R}^n)$ for $p = 1$ (see [63]), then by Theorems 4, 5, 7 and 8 we get

Corollary 10. *Suppose that $x_0 \in \mathbb{R}^n$, $\Omega \in L_s(S^{n-1})$, $1 < s \leq \infty$, is homogeneous of degree zero. Let $0 < \alpha < n$, $1 \leq p < \frac{n}{\alpha}$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. Let also, for $s' \leq p$, $p \neq 1$, the pair (φ_1, φ_2) satisfies condition (3.12) and for $q < s$ the pair (φ_1, φ_2) satisfies condition (3.13) and Ω satisfies conditions (a)–(c). Then the operator $\mu_{\Omega,\alpha}$ is bounded from $LM_{p,\varphi_1}^{\{x_0\}}$ to $LM_{q,\varphi_2}^{\{x_0\}}$ for $p > 1$ and from $LM_{1,\varphi_1}^{\{x_0\}}$ to $WLM_{q,\varphi_2}^{\{x_0\}}$ for $p = 1$.*

Corollary 11. *Suppose that $x_0 \in \mathbb{R}^n$, $\Omega \in L_s(S^{n-1})$, $1 < s \leq \infty$, is homogeneous of degree zero. Let $0 < \alpha < n$, $1 \leq p < \frac{n}{\alpha}$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. Let also, for $s' \leq p$, $p \neq 1$, the pair (φ_1, φ_2) satisfies conditions (2.5)–(2.6) and (3.16)–(3.17) and for $q < s$ the pair (φ_1, φ_2) satisfies conditions (2.5)–(2.6) and (3.18)–(3.19) and Ω satisfies conditions (a)–(c). Then the operator $\mu_{\Omega,\alpha}$ is bounded from $VLM_{p,\varphi_1}^{\{x_0\}}$ to $VLM_{q,\varphi_2}^{\{x_0\}}$ for $p > 1$ and from $VLM_{1,\varphi_1}^{\{x_0\}}$ to $WVLM_{q,\varphi_2}^{\{x_0\}}$ for $p = 1$.*

Corollary 12. *Suppose that $x_0 \in \mathbb{R}^n$, $\Omega \in L_s(S^{n-1})$, $1 < s \leq \infty$, is homogeneous of degree zero. Let $0 < \alpha < n$, $1 < p < \frac{n}{\alpha}$, $b \in LC_{p_2,\lambda}^{\{x_0\}}(\mathbb{R}^n)$, $0 \leq \lambda < \frac{1}{n}$, $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$, $\frac{1}{q_1} = \frac{1}{p_1} - \frac{\alpha}{n}$. Let also, for $s' \leq p$ the pair (φ_1, φ_2) satisfies condition (4.8) and for $q_1 < s$ the pair (φ_1, φ_2) satisfies condition (4.9) and Ω satisfies conditions (a)–(c). Then, the operator $[b, \mu_{\Omega,\alpha}]$ is bounded from $LM_{p_1,\varphi_1}^{\{x_0\}}$ to $LM_{q,\varphi_2}^{\{x_0\}}$.*

Corollary 13. *Suppose that $x_0 \in \mathbb{R}^n$, $\Omega \in L_s(S^{n-1})$, $1 < s \leq \infty$, is homogeneous of degree zero. Let $0 < \alpha < n$, $1 < p < \frac{n}{\alpha}$, $b \in LC_{p_2,\lambda}^{\{x_0\}}(\mathbb{R}^n)$, $0 \leq \lambda < \frac{1}{n}$, $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$, $\frac{1}{q_1} = \frac{1}{p_1} - \frac{\alpha}{n}$. Let also, for $s' \leq p$ the pair (φ_1, φ_2) satisfies conditions (2.5)–(2.6)–(4.11) and (4.12)–(4.10) and for $q_1 < s$ the pair (φ_1, φ_2) satisfies conditions (2.5)–(2.6)–(4.11) and (4.14)–(4.13) and Ω satisfies conditions (a)–(c). Then, the operator $[b, \mu_{\Omega,\alpha}]$ is bounded from $VLM_{p_1,\varphi_1}^{\{x_0\}}$ to $VLM_{q,\varphi_2}^{\{x_0\}}$.*

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